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The Controllability of the Coupled Wave Systems

Contrôlabilité de systèmes des ondes couplées

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Résumé

Dans cette thèse, nous étudions les théories étroitement liées du contrôle et les propriétés de la continuation unique, pour des équations et systèmes des ondes linéaires. Les résultats principaux proviennent des travaux de l'auteur:

1. Jingrui Niu. Simultaneous Control of Wave Systems. *SIAM J. Control Optim.*, 59(3):2381–2409, 2021

2. Pierre Lissy and Jingrui Niu. Controllability of a coupled wave system with a single control and different speeds. *preprint*, 2021

Dans (1), nous avons étudié la contrôlabilité simultanée des systèmes des ondes dans un domaine ouvert de $\mathbb{R}^d$. Nous obtenons un résultat de contrôlabilité partielle sur un espace co-dimensionnel fini pour des équations d'onde couplées par une seule fonction de contrôle. Pour la propriété de continuation unique des fonctions propres, nous avons donné un contre-exemple pour montrer que dans certaines métriques, la propriété de continuation unique n'est pas vraie. De plus, nous avons étudié différentes conditions pour garantir la propriété de continuation unique. Nous avons étudié également notre résultat au cas de coefficients constants et éventuellement de fonctions de contrôle multiples. Dans ce contexte, nous avons prouvé que la propriété de contrôlabilité est équivalente à une condition de rang de Kalman appropriée.

Dans (2), nous avons étudié un problème de contrôlabilité exact dans un domaine ouvert Ω de $\mathbb{R}^d$, pour un système des ondes couplées, avec des vitesses différentes et une seule commande agissant sur une sous-ensemble ouvert ω satisfaisant la condition de contrôle géométrique et sur une seule vitesse. Les actions pour les équations des ondes avec la deuxième vitesse sont obtenues par un terme de couplage. Tout d'abord, nous construisons des espaces d'états appropriés avec des conditions de compatibilité associées à la structure de couplage. Deuxièmement, dans ces espaces bien préparés, nous prouvons que le système des ondes couplées est exactement contrôlable si et seulement si la structure de couplage satisfait à une condition de rang de Kalman de l'opérateur.

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Chapter 1

Introduction(français)

1.1 Motivation

La contrôlabilité des équations d'onde est un sujet de recherche classique dans la théorie du contrôle et dans l'analyse des équations aux dérivées partielles. Il existe une grande littérature sur la contrôlabilité des équations des ondes linéaires. L'un des meilleurs résultats sur ce sujet a été obtenu par Bardos, Lebeau et Rauch dans leur article [10], où ils ont introduit la condition de contrôle géométrique et présenté l'application de l'analyse microlocale au sujet. On peut aussi se référer à l'article [14] de Burq et Gérard et à l'article [12] de Burq pour des améliorations ou des démonstrations plus simples. Ces résultats forment un contexte de base et fournissent également la stratégie principale pour nous d'étudier la contrôlabilité des équations des ondes.

Comme nous pouvons le voir, pour une équation des ondes scalaire, la contrôlabilité exacte est bien connue. Il existe une large littérature sur la contrôlabilité d'une équation des ondes scalaire à travers différentes approches telles que [10] en utilisant l'analyse microlocale comme nous l'avons mentionné précédemment, [38, 29] en utilisant des multiplicateurs, [25, 11] en utilisant des estimations de Carleman, ou une preuve complètement constructive [30], etc.

Bien que nous ayons maintenant une meilleure compréhension de la contrôlabilité d'une équation des ondes scalaire, la contrôlabilité des systèmes des ondes n'est toujours pas totalement comprise. A notre connaissance, la plupart des références concernent le cas de systèmes avec le même symbole principal. Alabau-Boussouira et Léautaud [5] ont étudié la contrôlabilité indirecte de deux équations des ondes couplées, dans lesquelles leur résultat de contrôlabilité a été établi en utilisant une méthode d'énergie multi-niveaux introduite dans [2], et également utilisé dans [3, 4]. Liard et Lissy [37], Lissy et Zuazua [40] ont étudié l'observabilité et la contrôlabilité des systèmes des ondes couplées sous la condition de rang de type

Kalman. De plus, nous pouvons trouver d'autres résultats de contrôlabilité pour les systèmes des ondes couplées, par exemple, Cui, Laurent et Wang [19] ont étudié l'observabilité des équations d'onde couplées par des termes d'ordre zéro ou du premier ordre sur une variété compacte. Cependant, lorsque l'on considère la contrôlabilité du système des ondes couplée à des vitesses différentes, il y a très peu de résultats.

Par contre, compte tenu de la contrôlabilité d'un système parabolique, nous constatons qu'il n'y a pas de différences entre le couplage avec la même vitesse et des vitesses différentes (par exemple, voir [6]). Cela nous motive également à étudier les résultats sur la contrôlabilité du système des ondes à différentes vitesses.

Dans cette thèse, le principal modal étudié est l'équation d'onde sous la forme suivante. Soit $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, un domaine borné et lisse. Pour les constantes positives α et β, soit $k_{ij}(x) : \Omega \to \mathbb{R}$, $1 \leq i,j \leq d$ des fonctions lisses qui satisfont:

$$k_{ij}(x) = k_{ji}(x), \alpha|\xi|^2 \leq \sum_{1\leq i,j\leq d} k_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2, \forall x \in \Omega, \forall \xi \in \mathbb{R}^d. \tag{1.1.1}$$

Supposons $K(x)$ est la matrice symétrique définie positive des coefficients $k_{ij}(x)$. De plus, nous définissons la fonction de densité $\kappa(x) = \frac{1}{\sqrt{\det(K(x))}}$. On définit également le Laplacien par $\Delta_K = \frac{1}{\kappa(x)} div(\kappa(x) K \nabla \cdot)$ sur Ω et l'opérateur d'Alembert $\Box_K = \partial_t^2 - \Delta_K$ sur $\mathbb{R}_t \times \Omega$. Nous considérons une équation d'onde non homogène avec un terme source f:

$$\Box_K u = f, \tag{1.1.2}$$

avec conditions initiales:

$$u|_{t=0} = u^0, \partial_t u|_{t=0} = u^1. \tag{1.1.3}$$

1.2 Généralités

Dans cette section, nous présenterons quelques aspects de base du problème de contrôle des équations d'onde. Nous supposons que ω est un sous-ensemble ouvert de Ω. Nous considérons le problème de contrôlabilité intérieure pour l'équation des ondes suivante:

$$\begin{cases} \Box_K u = f\mathbf{1}_\omega \text{ dans }]0,T[\times\Omega, \\ u = 0 \quad \text{ sur }]0,T[\times\partial\Omega, \\ u|_{t=0} = u^0(x), \quad \partial_t u|_{t=0} = u^1(x), \end{cases} \tag{1.2.1}$$

où f est une fonction de contrôle avec son support localisée dans le sous-domaine ω.

Il est bien connu que l'équation d'onde modélise de nombreux phénomènes physiques tels que les petites vibrations des corps élastiques et la propagation du son. Par exemple, (1.2.1) fournit une bonne approximation pour les vibrations de faible amplitude d'une corde élastique ou d'une membrane flexible occupant la région Ω au repos. La commande f représente alors une force localisée agissant sur la structure vibrante.

De plus, puisque l'équation d'onde est l'équation hyperbolique la plus pertinente. Par l'étude de l'équation d'onde, il nous aide à comprendre comment les propriétés des équations hyperboliques agissent sur les problèmes de contrôle.

Il est donc intéressant et important d'étudier la contrôlabilité de l'équation d'onde comme l'un des modèles fondamentaux de la mécanique du continuum et, en même temps, comme l'une des équations les plus représentatives de la théorie du contrôle des équations aux dérivées partielles.

1.2.1 Contrôlabilité

Dans cette section, nous présenterons plusieurs types différents de contrôlabilité pour l'équation d'onde (1.2.1).

Définition 1.2.1 (Contrôlabilité)**.** *Let $T > 0$.*

1. *(Contrôlabilité exacte) On dit que l'équation d'onde* (1.2.1) *est exactement contrôlable dans $H_0^1 \times L^2$ au temps T si pour toutes données initiales $(u^0, u^1) \in H_0^1 \times L^2$ et toutes données cibles $(\tilde{u}^0, \tilde{u}^1) \in H_0^1 \times L^2$, il existe un contrôle $f \in L^2(]0, T[\times\omega)$ tel que la solution de* (1.2.1) *avec les données initiales $(u|_{t=0}, \partial_t u|_{t=0}) = (u^0, u^1)$, satisfait $(u|_{t=T}, \partial_t u|_{t=T}) = (\tilde{u}^0, \tilde{u}^1)$.*

2. *(Contrôlabilité à zéro) On dit que l'équation d'onde* (1.2.1) *est contrôlable à zéro dans $H_0^1 \times L^2$ au temps T si pour toutes données initiales $(u^0, u^1) \in H_0^1 \times L^2$, il existe un contrôle $f \in L^2(]0, T[\times\omega)$ tel que la solution de* (1.2.1) *avec les données initiales $(u|_{t=0}, \partial_t u|_{t=0}) = (u^0, u^1)$, satisfait $(u|_{t=T}, \partial_t u|_{t=T}) = (0, 0)$.*

3. *(Contrôlabilité à partir de zéro) On dit que l'équation d'onde* (1.2.1) *est contrôlable à partir de zéro dans $H_0^1 \times L^2$ au temps T si pour toutes données cibles $(\tilde{u}^0, \tilde{u}^1) \in H_0^1 \times L^2$, il existe un contrôle $f \in L^2(]0, T[\times\omega)$ tel que la solution de* (1.2.1) *avec les données initiales $(u|_{t=0}, \partial_t u|_{t=0}) = (0, 0)$, satisfait $(u|_{t=T}, \partial_t u|_{t=T}) = (\tilde{u}^0, \tilde{u}^1)$.*

4. *(Contrôlabilité partielle) Soit Π un opérateur de projection défini dans $H_0^1 \times L^2$. On dit que l'équation d'onde* (1.2.1) *est $\Pi-$exactement contrôlable dans $H_0^1 \times L^2$ au temps T si pour toutes données initiales $(u^0, u^1) \in H_0^1 \times L^2$ et*

toutes données cibles $(\tilde{u}^0, \tilde{u}^1) \in H_0^1 \times L^2$*, il existe un contrôle* $f \in L^2(]0,T[\times\omega)$ *tel que la solution de* (1.2.1) *avec les données initiales* $(u|_{t=0}, \partial_t u|_{t=0}) = (u^0, u^1)$*, satisfait* $\Pi(u|_{t=T}, \partial_t u|_{t=T}) = \Pi(\tilde{u}^0, \tilde{u}^1)$*.*

Remarque 1.2.2. *En particulier, parce que l'équation d'onde est linéaire et réversible, la contrôlabilité exacte, la contrôlabilité à zéro et la contrôlabilité à partir de zéro sont équivalent (voir [17, Theorem 2.41]).*

1.2.2 Condition de Kalman

Dans cette section, nous rappelons quelques conditions de rang de Kalman introduites dans la littérature des systèmes paraboliques couplés. Tout d'abord, nous rappelons la condition de rang de Kalman pour la contrôlabilité des équations différentielles ordinaires autonomes linéaires (voir *par exemple* [27]).

Définition 1.2.3 (Condition de rang de Kalman)**.** *Soit* m*,* n *deux entiers positifs. Supposons* $A \in \mathcal{M}_n(\mathbb{R})$ *et* $B \in \mathcal{M}_{n,m}(\mathbb{R})$*. Nous introduisons la matrice de Kalman associée à* A *et* B *définie par* $[A|B] = [A^{n-1}B|\cdots|AB|B] \in \mathcal{M}_{n,nm}(\mathbb{R})$*. On dit que* (A, B) *satisfait la condition de rang de Kalman si* $[A|B]$ *est une matrice de plein rang.*

Cette condition de Kalman pour la contrôlabilité est introduite dans [28], qui est un critère pour un système linéaire autonome $\dot{x} = Ax + Bu$ avec un contrôle $u \in L^\infty(]T_0, T_1[, \mathbb{R}^m)$. De plus, nous remarquons que la condition de rang de Kalman est une condition équivalente pour la contrôlabilité du système linéaire autonome $\dot{x} = Ax + Bu$ (on peut se référer à [17, Remarque 1.17]).

Définition 1.2.4 (Opérateur de Kalman)**.** *Supposons que* $X \in \mathbb{R}^{n\times n}$ *and* $Y \in \mathbb{R}^{n\times m}$*. De plus, soit* $D \in \mathbb{R}^{n\times n}$ *une matrice diagonale. Alors, l'opérateur de Kalman associée à* $(-D\Delta+X, Y)$ *est une opérateur* $\mathscr{K} = [-D\Delta+X|Y] : D(\mathscr{K}) \subset (L^2)^{nm} \to (L^2)^n)$*, avec le domaine de l'opérateur* $\mathcal{D}(\mathscr{K}) = \{u \in (L^2)^{nm} : \mathscr{K}u \in (L^2)^n\}$*.*

Définition 1.2.5 (Condition de rang de l'opérateur de Kalman)**.** *On dit que l'opérateur de Kalman* $\mathscr{K}$ *satisfait la condition de rang de l'opérateur de Kalman si* $Ker(\mathscr{K}^*) = \{0\}$*.*

La condition de rang de l'opérateur Kalman peut être reformulée comme suit.

Proposition 1.2.6. *[6, Proposition 2.2] La condition de rang de l'opérateur Kalman est equivalent à la condition de rang de Kalman spectral suivante:*

$$rang[(\lambda D + X)|Y] = n, \forall \lambda \in \sigma(-\Delta).$$

En particulier, soit $C > 0$ *une constante et* $D = CId_n$*. Alors, La condition de rang de l'opérateur Kalman est equivalent à la condition de rang de Kalman donnée par Définition 1.2.3 (voir [6, Remark 1.2]).*

1.2.3 Méthode d'unicité de Hilbert

Pour l'équation (1.2.1), nous introduisons l'équation adjointe comme suit:

$$\begin{cases} \Box_K v = 0 \text{ dans }]0,T[\times\Omega, \\ v = 0 \quad \text{ sur }]0,T[\times\partial\Omega, \\ v|_{t=0} = v^0(x), \quad \partial_t v|_{t=0} = v^1(x), \end{cases} \tag{1.2.2}$$

Définition 1.2.7. *On dit qu'une équation d'onde homogène* (1.2.2) *est observable dans* $[0,T]\times\omega$ *s'il existe une constante* $C>0$ *telle que chaque solution* $v \in C^0(0,T,L^2)\cap C^1(0,T,H^{-1})$ *de l'équation d'onde homogène* (1.2.2) *satisfait à*

$$C\int_0^T\int_\omega |\kappa v|^2 dxdt \geq ||v^0||^2_{L^2} + ||v^1||^2_{H^{-1}}. \tag{1.2.3}$$

Ici, l'inégalité (1.2.3) est appelée l'inégalité d'observabilité pour l'équation adjointe.

Selon la méthode de l'unicité de Hilbert de J.-L. Lions [38], la propriété de contrôlabilité est équivalente à une inégalité d'observabilité pour le système adjoint.

Théorème 1.2.8. *L'équation d'onde* (1.2.1) *est contrôlable à zéro si et seulement si l'équation adjointe* (1.2.2) *est observable dans* $[0,T]\times\omega$.

L'idée de preuve de ce théorème est la méthode d'unicité de Hilbert, qui établit la dualité entre la contrôlabilité à zéro et l'observabilité. Nous définissons l'opérateur R par

$$R : f \in L^2(]0,T[\times\omega) \mapsto (u^0,u^1) \in H^1_0 \times L^2, \tag{1.2.4}$$

où u est la solution de (1.2.1) avec $(u|_{t=T}, \partial_t u|_{t=T}) = (0,0)$. D'autre part, nous définissons l'opérateur S par

$$S : (v^0,v^1) \in L^2\times H^{-1} \mapsto v\mathbf{1}_{]0,T[}(t)\mathbf{1}_\omega(x) \in L^2(]0,T[\times\omega), \tag{1.2.5}$$

où v résout l'équation adjointe (1.2.2). Par conséquent, la contrôlabilité à zéro est la surjectivité de l'opérateur R et l'observabilité est la coercitivité de l'opérateur S. Le Théorème 1.2.8 implique la dualité $R^* = S$.

Remarque 1.2.9. *La coercivité de S implique son injectivité, c'est-à-dire, un résultat de la continuation unique de* (1.2.2) *: si v résout* (1.2.2) *et s'annule dans* $[0,T]\times\omega$*, alors* $v\equiv 0$.

1.2.4 Condition du contrôle géométrique

Afin d'étudier l'inégalité d'observabilité, une méthode classique consiste à suivre le processus abstrait en trois étapes initialisé par Rauch et Taylor : [46](voir également [10]). Il peut être détaillé comme suit :

- Premièrement, obtenir l'information microlocale sur la région observable. Montrer par contradiction et on obtient différents types de convergence dans le sous-domaine $]0,T[\times\omega$ et le domaine $]0,T[\times\Omega$.

- Deuxièmement, utilisez la mesure de défaut microlocale (qui est due à Gérard [23] et Tartar [47]), ou le théorème de propagation des singulaties (voir [26, Section 18.1]) pour prouver une estimation d'observabilité faible :

$$||v^0||_{L^2}^2 + ||v^1||_{H^{-1}}^2 \leq C(\int_0^T \int_\omega |\kappa v|^2 dxdt + ||v^0||_{H^{-1}}^2 + ||v^1||_{H^{-2}}^2).$$

- Troisièmement, utilisez les propriétés de continuation unique des fonctions propres pour obtenir l'inégalité d'observabilité originale (1.2.3).

Pour les estimations à haute fréquence, une condition très naturelle consiste à supposer que l'ensemble de contrôle satisfait à la condition de contrôle géométrique (CCG).

Définition 1.2.10. *Pour un sous-ensemble ω et $T > 0$, nous dirons que la paire (ω, T, p_K) satisfait la condition de contrôle géométrique (CCG) si tout rayon bicharactéristique générale de p_K rencontre ω en un temps $t < T$, où p_K est le symbole principal de $\Box_K$.*

Nous donnerons la définition des bicharactéristiques dans la 1.3.1. Cette condition a été soulevée par Bardos, Lebeau et Rauch [9] lorsqu'ils ont considéré la contrôlabilité d'une équation scalaire à ondes et est maintenant devenue une hypothèse de base pour la contrôlabilité des équations à ondes. Dans [14], les auteurs montrent que la condition de contrôle géométrique est une condition nécessaire et suffisante pour la contrôlabilité exacte de l'équation d'onde avec conditions de Dirichlet et des contrôles aux limites continues.

1.2.5 Propriété de la continuation unique

Comme nous le savions, la propriété de la continuation unique n'implique pas la contrôlabilité en dimension infinie. En effet, par exemple, sur une variété riemannienne compacte, les valeurs propres du laplacien étant discrètes, le régime des basses fréquences est engendré par un nombre fini de fonctions propres du laplacien. C'est essentiellement l'idée de l'argument unicité-compacité dans l'article de

Bardos, Lebeau, et Rauch [38]. Cet argument ramène l'observabilité du régime des basses fréquences à la propriété de la continuation unique des fonctions propres du laplacien. C'est-à-dire, si u satisfait l'équation,

$$-\Delta u = \lambda u, \lambda \in \mathbb{C} \tag{1.2.6}$$

et si $u|_\omega = 0$, a-t-on $u \equiv 0$ dans Ω?

Lorsque Δ est un opérateur différentiel à coefficients analytiques, Holmgren a montré l'unicité de solution parmi les distributions. Le premier effort pour supprimer l'analyticité est dû à Carleman [16], qui a montré l'unicité en supposant que les caratéristiques de l'équation sont simples. Il y a beaucoup de litterature sur l'inégalité de Craleman, par exemple, voir [**?**, **?**].

1.3 Mesure de défaut pour l'équation des ondes

1.3.1 Préliminaires géométriques

Soit $B = \{y \in \mathbb{R}^d : |y| < 1\}$ la boule unité de $\mathbb{R}^d$ et localement on identifie $M = \Omega \times \mathbb{R}_t$ avec $[0, 1[\times B$. Pour $z \in \overline{M} = \overline{\Omega} \times \mathbb{R}_t$, on note $z = (x, y)$,où $x \in [0, 1[$ et $y \in B$. De plus, $z \in \partial M = \partial\Omega \times \mathbb{R}_t$ si et seulement si $z = (0, y)$. Soit $R = R(x, y, D_y)$ un opérateur pseudo-différentiel scalaire (C^∞) tangentiel classique de degré 2, auto-joint, défini au voisinage de $[0, 1] \times B$, de symbole principal réel $r(x, y, \eta)$, on définit les fonctions r_0 et r_1 par

$$r(x, y, \eta) = r_0(y, \eta) + x r_1(y, \eta) + O(x^2).$$

On supposera que la fonction homogène de degré 2 en η, $r(x, y, \eta)$ vérifie

$$\frac{\partial r}{\partial \eta} \neq 0 \text{ pour } (x, y) \in [0, 1[\times B \text{ et } \eta \neq 0.$$

Soitent également $Q_0(x, y, D_y)$ et $Q_1(x, y, D_y)$ des opérateurs pseudo-différentiels tangentiels classiques définis au voisinage de de $[0, 1] \times B$, de degrés respectifs 0 et 1, de symboles principaux q_0 et q_1. On note P l'opérateur de degré 2:

$$P = (\partial_x^2 + R) + Q_0 \partial_x + Q_1$$

Le symbole principal p de P est scalaire et vaut $p = -\xi^2 + r(x, y, \eta)$. Donc, on décompose $T^*\partial M$ en l'union disjointe $\mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$, où

$$\mathcal{E} = \{r_0 < 0\}, \mathcal{G} = \{r_0 = 0\}, \mathcal{H} = \{r_0 > 0\}.$$

On dit que $\rho \in \mathcal{E}$ est elliptique, $\rho \in \mathcal{G}$ est glissant et $\rho \in \mathcal{H}$ est hyperbolique. On note Char(P) la variété caractéristique de P par

$$Char(P) = \{(x, y, \xi, \eta) \in T^*\mathbb{R}^{d+1}|_{\overline{M}} : \xi^2 = r(x, y, \xi, \eta)\}.$$

Pour l'ensemble glissant $\mathcal{G}$, on a la décomposition $\mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j$, avec

$$\begin{aligned}
\mathcal{G}^2 &= \{(y, \eta) : r_0(y, \eta) = 0, r_1(y, \eta) \neq 0\}, \\
\mathcal{G}^3 &= \{(y, \eta) : r_0(y, \eta) = 0, r_1(y, \eta) = 0, H_{r_0}(r_1) \neq 0\}, \\
&\vdots \\
\mathcal{G}^{k+3} &= \{(y, \eta) : r_0(y, \eta) = 0, H_{r_0}^j(r_1) = 0, \forall j \leq k, H_{r_0}^{k+1}(r_1) \neq 0\}, \\
&\vdots \\
\mathcal{G}^{\infty} &= \{(y, \eta) : r_0(y, \eta) = 0, H_{r_0}^j(r_1) = 0, \forall j\}.
\end{aligned}$$

Ici, H_{r_0} est le champ de vecteurs hamiltonien de r_0. De plus, pour $\mathcal{G}^2$, on note $\mathcal{G}^{2,\pm} = \{(y, \eta) : r_0(y, \eta) = 0, \pm r_1(y, \eta) > 0\}$. Alors, $\mathcal{G}^2 = \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}$. On dit que $\rho \in \mathcal{G}^{2,-}$ est strictement glissant et $\rho \in \mathcal{G}^{2,+}$ est diffractif. Pour $\rho \in \mathcal{G}^j$, on dit que ρ est glisant d'ordre j.

Définition 1.3.1. *On dit que Ω n'a pas de contact d'ordre infini avec ses tangentes si il existe $N \in \mathbb{N}$ telle que $\mathcal{G} = \bigcup_{j=2}^{N} \mathcal{G}^j$.*

Définition 1.3.2. *On appelle bicaractéristique généralisée toute application continue γ, de $\mathbb{R}$ dans T_b^*M telle qu'en dehors d'un ensemble de points isolés I, $\gamma(s) \in T^*M \cup \mathcal{G}$, si $s \in I$, on a $\gamma(s) \in \mathcal{H}$ et si $s \notin I$, γ est différentiable avec*

1. $\frac{d\gamma}{ds}(s) = H_p(\gamma(s))$ *si* $\gamma(s) \in T^*M \cup \mathcal{G}^{2,+}$
2. $\frac{d\gamma}{ds}(s) = H_{-r_0}(\gamma(s))$ *si* $\gamma(s) \in \mathcal{G} \backslash \mathcal{G}^{2,+}$.

Remarque 1.3.3. *Il est classique que les définitions que nous avons exprimées en coordonnées sont intrinsèques et que si Ω n'a pas contact d'ordre infini avec ses tangentes, par tout point ρ_0 il passe une et une seule bicaractéristique généralisée telle que $\gamma(0) = \rho_0$. Voir [42, 43].*

Pour plus de détails, voir [15] and [13].

1.3.2 Mesure de défaut

Dans cette section, nous allons donner deux approches pour construire la mesure de défaut. La première est basée sur l'article de Gérard et Leichtnam [24] pour l'équation de Helmoltz et Burq [13] pour l'équation d'onde. L'autre suit l'idée de l'article [31] de Lebeau et nous nous appuyons sur l'article [15] de Burq et Lebeau pour la mise en place des systèmes d'onde. Pour la première, on considère $(u^k)_{k\in\mathbb{N}} \subset \left(L^2_{loc}(\mathbb{R}^+; L^2(\Omega))\right)^n$ une suite bornée d'élements de $\left(L^2_{loc}(\mathbb{R}^+; L^2(\Omega))\right)^n$, qui satisfait

$$\begin{cases} Pu^k = o(1)_{H^{-1}}, \\ u^k|_{\partial M} = 0. \end{cases} \tag{1.3.1}$$

On suppose que la suite (u^k) converge faiblement vers 0 et on note $\underline{u}_k \subset \left(L^2_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^d))\right)^n$ le prolongement par 0 de u^k à l'extérieur de l'ouvert M. Suivant la [13, Section 1], nous avons l'existence de la mesure de défaut microlocale comme suit :

Proposition 1.3.4. *On peut donc, quitte à extraire une sous-suite lui associer une mesure positive sur $S^*((\mathbb{R}^+ \times \mathbb{R}^d))$ $\underline{\mu}$, vérifiant pour tout $A \in \underline{\mathcal{A}}$*

$$\lim_{k\to\infty} (A\underline{u}^k, \underline{u}^k)_{L^2} = \langle \underline{\mu}, \sigma(A)\rangle, \tag{1.3.2}$$

où $\underline{\mathcal{A}}$ est une espace des matrices $n \times n$ d'opérateurs pseudo-différentiels classiques d'ordre 0, à support compact dans $\mathbb{R}^+ \times \mathbb{R}^d$ et $\sigma(A)$ est le symbole principal d'opérateur A, qui est une matrice de fonctions lisses, homogènes d'ordre 0 dans la variable ξ, c'est-à-dire une fonction sur $S^*((\mathbb{R}^+ \times \mathbb{R}^d))$.

D'après [13, Théorème 15], nous avons la proposition suivante.

Proposition 1.3.5. *La mesure de défaut $\underline{\mu}$ vérifie les propriétés suivantes:*

- *La support de la mesure $\underline{\mu}$ est inclus dans l'intersection de la variété caractéristique de l'équation des ondes avec $\mathbb{R}^+ \times \overline{\Omega}$:*

$$supp(\underline{\mu}) \subset Char(P) = \{(t, x, \tau, \xi); x \in \overline{M}, \tau^2 = |\xi|^2_x\}. \tag{1.3.3}$$

- *La mesure $\underline{\mu}$ ne charge pas l'ensemble hyperbolique dans ∂M :*

$$\underline{\mu}(\mathcal{H}) = 0.$$

- *En particulier, si $n = 1$, la mesure scalaire $\underline{\mu}$ est invariante le long du flot bicaractéristique généralisé.*

D'autre part, on note $\mathcal{A}$ une espace des matrices $n \times n$ d'opérateurs A de la forme $A = A_i + A_t$ où A_i est un opérateur pseudo-différentiel classique d'ordre 0, à support compact dans M (i.e, vérifiant $A_i = \varphi A_i \varphi$ pour un $\varphi \in C_0^\infty(M)$) et où A_t est un opérateur pseudo-différentiel tangentiel classique d'ordre 0, à support compact dans $\overline{M}$ (i.e, vérifiant $A_t = \varphi A_t \varphi$ pour un $\varphi \in C^\infty(\overline{M})$).

Remarque 1.3.6. *On note le fibre de cotangente compressée de Melrose par* ${}^bT^*\overline{M}$ *et l'appication canonique* j:$T^*\overline{M} \mapsto {}^bT^*\overline{M}$*, défini par*

$$j(x, y, \xi, \eta) = (x, y, x\xi, \eta).$$

On pose

$$Z = j(Char(P)), \quad \hat{Z} = Z \cup j(T^*\overline{M}|_{x=0}),$$

et

$$S\hat{Z} = (\hat{Z} \backslash \overline{M})/\mathbb{R}^*_+, \quad SZ = (Z \backslash \overline{M})/\mathbb{R}^*_+.$$

Remarque 1.3.7. $S\hat{Z}$ *et* SZ *sont les espaces quotients sphérique et des espaces métriques localement compacts.*

Pour $A \in \mathcal{A}$, avec le symbole principal $a = \sigma(A)$, on définit

$$\kappa(a)(\rho) = a(j^{-1}(\rho)), \forall \rho \in {}^bT^*\overline{M}.$$

Donc, on obtient l'ensemble $\mathcal{K} = \{\kappa(a) : a = \sigma(A), A \in \mathcal{A}\} \subset C^0(S\hat{Z}; End(\mathbb{C}^n))$. On notera $\mathcal{M}^+$ l'espace des mesures boréliennes μ sur $S\hat{Z}$, à valeurs hermitiennes positives sur $\mathbb{C}^n$. Une mesure μ de $\mathcal{M}^+$ est donc un élément du dual de l'espace $C^0_0(S\hat{Z}; End(\mathbb{C}^n))$ qui vérifie

$$\langle \mu, a \rangle \geq 0, \forall a \in C^0(S\hat{Z}; End^+(\mathbb{C}^n)), \forall \mu \in \mathcal{M}^+,$$

où $End^+(\mathbb{C}^n)$ désigne l'ensemble des matrices $n \times n$ hermitiennes positives.

Proposition 1.3.8. *Quitte à extraire une sous-suite de la suite* (u^k)*, il existe une mesure* $\mu \in \mathcal{M}^+$ *telle que*

$$\forall A \in \mathcal{A}, \quad \lim_{k \to \infty} (Au^k, u^k)_{L^2} = \langle \mu, \kappa(\sigma(A)) \rangle. \tag{1.3.4}$$

Pour plus de détails, voir [15]. On considére S une hypersurface transverse à le flot Melrose-Sjöstrand sur SZ. Alors localement, $SZ = \mathbb{R}_s \times S$ où s est le paramètre bien choisi le long de le flot.

Lemme 1.3.9. *La mesure* μ *vérifie les propriétés suivantes: La support de la mesure* μ *est inclus dans* SZ *et il existe une fonction*

$$(s, z) \in \mathbb{R}_s \times S \mapsto M(s, z) \in \mathbb{C}^n$$

qui est continue μ*-presque partout telle que la mesure* $\mathcal{P}^*\mu = M^*\mu M$ *défini pour* $a \in C^0(SZ)$ *par*

$$\langle M^*\mu M, a \rangle = \langle \mu, MaM^* \rangle$$

vérifie

$$\frac{d}{ds}\mathcal{P}^*\mu = 0.$$

On dit que la mesure μ est invariante le long du flot associé à M. De plus, la fonction M est continue et le long de toute bicaractéristique généralisée la matrice M est solution d'une équation différentielle dont les coefficients peuvent être explicitement calculés en termes de géométrie et des différents termes de l'opérateur P.

Pour l'équation différentielle de M, on peut voir [15, Section 3.2].

1.4 La contrôlabilité d'une équation d'onde scalaire

Dans cette section, nous donnons une preuve schématique de la contrôlabilité d'une équation d'onde scalaire telle que nous l'avons introduite en (1.2.1):

$$\begin{cases} \Box_K u = f\mathbf{1}_\omega \text{ dans }]0,T[\times\Omega, \\ u = 0 \quad \text{sur }]0,T[\times\partial\Omega, \\ u|_{t=0} = u^0(x), \quad \partial_t u|_{t=0} = u^1(x), \end{cases} \tag{1.4.1}$$

où nous supposons que $f \in L^2(]0,T[\times\omega)$ et les données initiales $(u^0,u^1) \in H_0^1(\Omega)\times L^2(\Omega)$. Nous considérons la contrôlabilité à zéro. La preuve est basée sur les trois étapes suivantes :

1. (Observabilité) En appliquant la méthode d'unicité de Hilbert, la propriété de contrôlabilité est équivalente à une inégalité d'observabilité pour le système adjoint. Ici, nous devons seulement prouver : $\exists C > 0$ tel que pour toutes solutions de l'équation adjointe :

$$\begin{cases} \Box_K v = 0 \text{ dans }]0,T[\times\Omega, \\ v = 0 \quad \text{sur }]0,T[\times\partial\Omega, \\ v|_{t=0} = v^0(x), \quad \partial_t v|_{t=0} = v^1(x), \end{cases} \tag{1.4.2}$$

   on a

$$||v^0||_{L^2}^2 + ||v^1||_{H^{-1}}^2 \le C\int_0^T\int_\omega |v|^2 dxdt. \tag{1.4.3}$$

2. (Estimations à haute fréquence) Nous établissons d'abord une inégalité d'observabilité faible comme suit :

$$||v^0||_{L^2}^2 + ||v^1||_{H^{-1}}^2 \le C\left(\int_0^T\int_\omega |v|^2 dxdt + ||v^0||_{H^{-1}}^2 + ||v^1||_{H^{-2}}^2\right). \tag{1.4.4}$$

Nous prouvons cette inégalité par l'argument de contradiction. Supposons que l'inégalité (1.4.2) soit fausse, il existe une suite $(v^{k,0}, v^{k,1})_{k\in\mathbb{N}}$ dans $L^2 \times H^{-1}$ telle que

$$||v^{k,0}||^2_{L^2} + ||v^{k,1}||^2_{H^{-1}} = 1, \tag{1.4.5}$$

$$||v^{k,0}||^2_{H^{-1}} + ||v^{k,1}||^2_{H^{-2}} \to 0, k \to \infty \tag{1.4.6}$$

$$\int_0^T \int_\omega |v^k|^2 dxdt \to 0, k \to \infty \tag{1.4.7}$$

où v^k est la solution de (1.4.2) avec les données initiales $(v^{k,0}, v^{k,1})$. Par conséquent, il existe une mesure de défaut microlocale μ associée à la suite bornée v^k. D'après la section précédente, nous savons que μ est invariant le long du flot bicaractéristique généralisé. De plus, nous savons que $\mu|_{]0,T[\times\omega} = 0$ par (1.4.7). Par conséquent, on obtient $\mu \equiv 0$. En combinant avec la loi de conservation de l'énergie de l'équation d'onde homogène (1.4.2), il y a une contradiction avec l'hypothèse (1.4.5). Par conséquent, nous prouvons l'inégalité d'observabilité faible (1.4.4).

3. (Estimations à basse fréquence) Nous utilisons l'inégalité d'observabilité faible (1.4.4) pour montrer l'observabilité originale (1.4.3). Nous argumentons également par contradiction. Supposons que (1.4.3) soit fausse, alors, il existe une séquence $(v^{k,0}, v^{k,1})_{k\in\mathbb{N}}$ dans $L^2 \times H^{-1}$ telle que

$$||v^{k,0}||^2_{L^2} + ||v^{k,1}||^2_{H^{-1}} = 1, \tag{1.4.8}$$

$$\int_0^T \int_\omega |v^k|^2 dxdt \to 0, k \to \infty \tag{1.4.9}$$

où v^k est la solution de (1.4.2) avec les données initiales $(v^{k,0}, v^{k,1})$. D'après l'inégalité d'observabilité faible, on a

$$1 = ||v^{k,0}||^2_{L^2} + ||v^{k,1}||^2_{H^{-1}} \le C \left(\int_0^T \int_\omega |v^k|^2 dxdt + ||v^{k,0}||^2_{H^{-1}+}||v^{k,1}||^2_{H^{-2}} \right). \tag{1.4.10}$$

Supposons que $(v^{k,0}, v^{k,1}) \rightharpoonup (v^0, v^1)$ in $L^2 \times H^{-1}$ et v est la solution de l'équation adjointe (1.4.2) avec les données initiales (v^0, v^1). Puisque $L^2 \times H^{-1} \mapsto H^{-1} \times H^{-2}$ est compact, nous savons que $||v^{k,0}||^2_{H^{-1}} + ||v^{k,1}||^2_{H^{-2}} \to ||v^0||^2_{H^{-1}} + ||v^1||^2_{H^{-2}}$. En conséquence, si k tend vers l'infini, on obtient

$$1 \le C \left(||v^0||^2_{H^{-1}+}||v^1||^2_{H^{-2}} \right). \tag{1.4.11}$$

On note

$$\mathcal{N}(T) = \{(w^0, w^1) \in L^2 \times H^{-1} : w(t,x) = 0, \text{ pour } t \in]0,T[, x \in \omega\}. \tag{1.4.12}$$

Ici w est une solution de l'équation adjointe (1.4.2) avec les données initiales (w^0, w^1). Par conséquent, $(v^0, v^1) \in \mathcal{N}(T)$. Ensuite, nous prouvons que $\mathcal{N}(T) = \{0\}$. D'après (1.4.4), nous savons que $\mathcal{N}(T)$ a une dimension finie. On note $\mathscr{A} = \begin{pmatrix} 0 & 1 \\ -\Delta_K & 0 \end{pmatrix}$. Alors $\mathcal{N}(T)$ est stable sous l'application de $\mathscr{A}$. Par conséquent, $\mathcal{N}(T)$ contient un vecteur propre de $\mathscr{A}$, c'est-à-dire que $\exists \lambda \in \mathbb{C}$ et $(\phi_0, \phi_1) \in H_0^1 \times L^2$ tel que

$$\begin{cases} \mathscr{A} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \lambda \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}, \text{ dans } \Omega, \\ \phi_0 = 0, \text{ dans } \omega. \end{cases} \tag{1.4.13}$$

Ceci est équivalent à : pour $\lambda \in \mathbb{C}$ et $\phi_0 \in H_0^1$

$$\begin{cases} -\Delta \phi_0 = \lambda^2 \phi_0, \text{ dans } \Omega, \\ \phi_0 = 0, \text{ dans } \omega. \end{cases} \tag{1.4.14}$$

Il s'agit d'un problème classique de continuation unique. En utilisant les estimations de Carleman (voir [16]), nous obtenons que $\phi_0 \equiv 0$. Par conséquent, nous savons que $\mathcal{N}(T) = \{0\}$. Par conséquent, nous avons $(v^0, v^1) = (0, 0)$, ce qui est une contradiction avec l'hypothèse (1.4.11). Par conséquent, nous prouvons l'inégalité d'observabilité (1.4.3).

1.5 Les systèmes des ondes couplées

1.5.1 Couplé à la fonction de contrôle

Dans cette section, on considére le problème de contrôlabilité simultanée d'un système d'onde avec différentes vitesses. On peut trouver ce résultat dans [44].

Un modèle simple

Nous présentons d'abord un exemple simple comme suit :

$$\begin{cases} (\partial_t^2 - \Delta) u_1 = f \mathbf{1}_{]0,T[}(t) \mathbf{1}_\omega(x) \\ (\partial_t^2 - 2\Delta) u_2 = f \mathbf{1}_{]0,T[}(t) \mathbf{1}_\omega(x) \\ u_j = 0 \quad \text{sur }]0,T[\times \partial\Omega, j = 1, 2, \\ u_j(0, x) = u_j^0(x) \in H_0^1, \quad \partial_t u_j(0, x) = u_j^1(x) \in L^2, j = 1, 2. \end{cases} \tag{1.5.1}$$

Remarquez que ces deux équations d'onde ont des vitesses différentes et que nous utilisons la même fonction de contrôle $f \in L^2(]0, T[\times \omega)$ pour contrôler les deux équations en même temps. Pour l'exemple (1.5.1), en appliquant la méthode d'unicité de Hilbert, nous devons seulement prouver une inégalité d'observabilité

$$\sum_{i=1}^{2}(||v_i^0||_{L^2}^2 + ||v_i^1||_{H^{-1}}^2) \leq C \int_0^T \int_\omega |v_1 + v_2|^2 dxdt \tag{1.5.2}$$

pour les solutions (v_1, v_2) du système adjoint avec les données initiales (v_i^0, v_i^1) :

$$\begin{cases} (\partial_t^2 - \Delta)v_1 = 0 \\ (\partial_t^2 - 2\Delta)v_2 = 0 \end{cases} \tag{1.5.3}$$

Pour prouver l'inégalité (1.5.3), nous estimons d'abord le régime à haute fréquence. Puisque les deux équations d'onde ont des vitesses différentes, alors les manifolds caractéristiques sont disjoints, ce qui implique que $||v_1 + v_2||_{L^2}^2 \approx ||v_1||_{L^2}^2 + ||v_2||_{L^2}^2$ dans le régime haute fréquence. Avec l'application de la mesure de défaut, nous savons que pour les hautes fréquences, observer la somme $v_1 + v_2$ est presque équivalent à observer v_1 et v_2. Ensuite, on s'intéresse au régime des basses fréquences. Il est équivalent de considérer un problème de continuation unique pour les fonctions propres comme suit : seules les solutions nulles satisfont

$$\begin{cases} -\Delta\phi_1 = \lambda\phi_1 \text{ dans } \Omega, \\ -2\Delta\phi_2 = \lambda\phi_2 \text{ dans } \Omega, \\ \phi_1 + \phi_2 = 0 \text{ dans } \omega. \end{cases} \tag{1.5.4}$$

Dans cet exemple, cette propriété est facile à prouver. Comme les fonctions propres du laplacien sont analytiques, nous savons que $\phi_1 + \phi_2 \equiv 0$ dans tout le domaine Ω. Ensuite, en additionnant deux équations, on obtient que $\Delta\phi_2 = 0$. En combinant avec la condition de Dirichlet, nous savons que $\phi_2 \equiv 0$, ce qui implique que $\phi_1 = -\phi_2 \equiv 0$. Par conséquent, on peut prouver ce problème de contrôle simultané. Par conséquent, nous concluons trois caractéristiques de ce type de problème :

1. Les équations d'onde ont des vitesses différentes alors que nous utilisons la même fonction de contrôle pour contrôler toutes ces équations en même temps.

2. En considérant l'inégalité d'observabilité, nous utilisons la norme localisée (restreinte dans le sous-domaine ω) de la somme des solutions pour contrôler la norme d'énergie totale des données initiales.

3. Nous avons besoin d'une propriété de continuation unique pour les fonctions propres associées au système d'onde.

Contrôle simultané des systèmes d'ondes

Dans mon article [44], on considère la contrôlabilité exacte sur un domaine ouvert Ω de systèmes des ondes avec des vitesses différentes, couplés par une seule fonction de contrôle agissant sur un sous-ensemble ouvert ω. Pour être plus précis, on considère la contrôlabilité intérieure simultanée pour le système des ondes suivant :

$$\begin{cases} \Box_{K_1} u_1 = b_1 f \mathbf{1}_{]0,T[}(t)\mathbf{1}_\omega(x) \text{ dans }]0,T[\times\Omega, \\ \Box_{K_2} u_2 = b_2 f \mathbf{1}_{]0,T[}(t)\mathbf{1}_\omega(x) \text{ dans }]0,T[\times\Omega, \\ \vdots \\ \Box_{K_n} u_n = b_n f \mathbf{1}_{]0,T[}(t)\mathbf{1}_\omega(x) \text{ dans }]0,T[\times\Omega, \\ u_j = 0 \quad \text{sur }]0,T[\times\partial\Omega, 1 \leq j \leq n, \\ u_j(0,x) = u_j^0(x), \quad \partial_t u_j(0,x) = u_j^1(x), 1 \leq j \leq n. \end{cases} \tag{1.5.5}$$

Ici, nous choisissons $K_i (1 \leq i \leq n)$ pour être n différentes matrices symétriques définies positives, ce qui est une généralisation de n différentes vitesses d'onde de différentes métriques constantes. En outre, il est également important que la même fonction de contrôle f apparaît dans toutes les équations. $\{b_i\}_{1\leq i\leq n}$ sont n coefficients constants non nuls. Nous pourrions considérer cet exemple comme un cas particulier où le couplage n'apparaît que dans la fonction de contrôle. Pour ce système, nous sommes en mesure de prouver le résultat de contrôlabilité partielle comme suit :

Théorème 1.5.1. *Pour $T > 0$, supposons que :*

1. *(ω, T, p_{K_i}) satisfait CCG, $i = 1, 2, \cdots, n$,*
2. *$K_1 > K_2 > \cdots > K_n$ dans ω,*
3. *Ω n'a pas de contact d'ordre infini avec ses tangentes.*

Alors, il existe un sous-espace de dimension finie $E \subset (H_0^1(\Omega) \times L^2(\Omega))^n$ tel que le système (1.5.5) *est $\mathbb{P}-$exactement contrôlable, où $\mathbb{P}$ est le projecteur orthogonal sur $E^\perp$.*

Comme nous l'avons présenté précédemment, afin d'étudier les basses fréquences, nous devons introduire la notion de continuation unique des fonctions propres.

Définition 1.5.2. *On dit que le système* (1.5.5) *satisfait à la propriété de continuation unique des fonctions propres si la propriété suivante est vérifiée : $\forall \lambda \in \mathbb{C}$, la seule solution $(\phi_1, \cdots, \phi_n) \in (H_0^1(\Omega))^n$ de*

$$\begin{cases} -\Delta_{K_1}\phi_1 = \lambda^2\phi_1 \ \textit{dans } \Omega, \\ -\Delta_{K_2}\phi_2 = \lambda^2\phi_2 \ \textit{dans } \Omega, \\ \cdots \\ -\Delta_{K_n}\phi_n = \lambda^2\phi_n \ \textit{dans } \Omega, \\ b_1\kappa_1\phi_1 + \cdots + b_n\kappa_n\phi_n = 0 \ \textit{dans } \omega, \end{cases}$$

est la solution zéro $(\phi_1, \cdots, \phi_n) \equiv 0$.

Donc, on a la contrôlabilité exacte comme suit

Théorème 1.5.3. *Pour*$T > 0$, *supposons que :*

1. (ω, T, p_{K_i}) *satisfait CCG,* $i = 1, 2, \cdots, n$,
2. $K_1 > K_2 > \cdots > K_n$ *dans* ω,
3. Ω *n'a pas de contact d'ordre infini avec ses tangentes,*
4. *The system* (1.5.5) *satisfait à la propriété de continuation unique des fonctions propres.*

Alors, le système (1.5.5) *est exactement contrôlable dans* $(H_0^1(\Omega) \times L^2(\Omega))^n$.

Comme nous l'avons présenté dans la section précédente, nous prouvons ce théorème par une procédure similaire. D'abord, nous appliquons la méthode d'unicité de Hilbert, et obtiendrons l'inégalité d'observabilité : $\exists C > 0$ telle que pour toute solution du système adjoint :

$$\begin{cases} \square_{K_1} v_1 = 0 \text{ dans }]0, T[\times \Omega, \\ \square_{K_2} v_2 = 0 \text{ dans }]0, T[\times \Omega, \\ \vdots \\ \square_{K_n} v_n = 0 \text{ dans }]0, T[\times \Omega, \\ v_j = 0 \quad \text{sur }]0, T[\times \partial\Omega, 1 \leq j \leq n, \\ (v_1(0,x), \partial_t v_1(0,x), \cdots, v_n(0,x)\partial_t v_n(0,x)) = V^0, \end{cases} \tag{1.5.6}$$

où $V^0 \in (L^2 \times H^{-1})^n$, nous avons :

$$C \int_0^T \int_\omega |b_1 \kappa_1 v_1 + \cdots + b_n \kappa_n v_n|^2 dxdt \geq ||V^0||^2_{(L^2 \times H^{-1})^n}. \tag{1.5.7}$$

Il nous suffit alors de prouver cette inégalité d'observabilité (1.5.7). En regardant la haute fréquence, nous prouvons une estimation d'observabilité faible :

$$||V^0||^2_{(L^2 \times H^{-1})^n} \leq C \left(\int_0^T \int_\omega |\sum_{j=1}^n b_j \kappa_j v_j|^2 dxdt + ||V^0||^2_{(H^{-1} \times H^{-2})^n} \right). \tag{1.5.8}$$

En supposant que l'inégalité ci-dessus soit fausse, nous pourrions obtenir une séquence $(V^{0,k})_{k \in \mathbb{N}}$ telle que:

$$||V^{0,k}||^2_{(L^2 \times H^{-1})^n} = 1, \tag{1.5.9}$$

$$\int_0^T \int_\omega |b_1\kappa_1 v_1^k + \cdots + b_n\kappa_n v_n^k|^2 dxdt \to 0, k \to \infty, \tag{1.5.10}$$

and

$$||V^{0,k}||^2_{(H^{-1}\times H^{-2})^n} \to 0, k \to \infty. \tag{1.5.11}$$

Nous utilisons ici $v_i^k (1 \leq i \leq n)$ pour désigner la solution du système (1.5.6) avec les données initiales $V^{0,k}$. Puisque nous avons l'hypothèse 2, nous savons que les variétés caractéristiques de chaque équation d'onde sont disjointes, ce qui implique que

$$\int_0^T \int_\omega |b_1\kappa_1 v_1^k + \cdots + b_n\kappa_n v_n^k|^2 dxdt \approx \sum_{i=1}^n \int_0^T \int_\omega |b_i\kappa_i v_i^k|^2 dxdt \tag{1.5.12}$$

Par conséquent, nous savons que chaque mesure de défaut μ_i associée à v_i^k est nulle par l'application de la propagation des mesures de défaut et CCG. Ceci fournit une contradiction avec $||V^{0,k}||^2_{(L^2\times H^{-1})^n} = 1$. Ensuite, nous combinons l'hypothèse (4), nous savons que l'inégalité d'observabilité est vraie. Cela nous donne le résultat de la contrôlabilité exacte du système (1.5.5).

Quelques résultats sur les propriétés de continuation unique

Comme nous pouvons le voir dans l'exemple simple, les propriétés de continuation unique définies dans la Définition 1.5.2 sont vraies pour les métriques à coefficient constant. Mais nous pouvons aussi construire un contre-exemple tel que cette propriété de continuation unique ne tienne pas. En dimension 1, nous supposons que la métrique $g = c(x)dx^2$. Alors, $\Delta_g = \frac{1}{c}\frac{d^2}{dx^2} - \frac{c'}{2c^2}\frac{d}{dx}$. Fixons l'intervalle ouvert $]0, \pi[$ et le sous-intervalle $]a, b[\subset]0, \pi[$, $(a > \frac{\pi}{2})$. Nous considérons maintenant le problème de la continuation unique :

$$\begin{cases} u_1'' = -\lambda^2 u_1, \\ \Delta_g u_2 = -\lambda^2 u_2, \\ u_1 + u_2 = 0 \text{ in }]a, b[, \\ u_1, u_2 \in H_0^1(]0, \pi[). \end{cases} \tag{1.5.13}$$

Nous avons le résultat suivant :

Théorème 1.5.4. *Il existe une métrique riemannienne lisse $g = c(x)dx^2$, et deux fonctions propres u_1, u_2 de Δ_g et $\frac{d^2}{dx^2}$ sur $]0, \pi[$ associée à la valeur propre 1 telle que $u_1 + u_2 = 0$, dans $]a, b[\subset]0, \pi[$ et $u_1 + u_2 \not\equiv 0$ dans $]0, \pi[$.*

Les lecteurs peuvent trouver la construction détaillée de ce contre-exemple dans la section 3.5. En regardant le système 1.5.13, nous considérons l'intersection du spectre de deux Laplaciens avec des métriques différentes. Définissons l'espace de toutes les métriques lisses sur l'intervalle ouvert $]0,\pi[$ par $\mathcal{M}^1$. Nous prouvons la proposition suivante :

Proposition 1.5.5. *En dimension* 1*, supposons que nous fixions le laplacien* $\Delta = \frac{d^2}{dx^2}$ *dans* $]0,\pi[$ *avec son spectre* $\sigma(\Delta)$*. Alors l'ensemble* $\mathcal{G}_{uc} = \{g \in \mathcal{M}^1 : \sigma(\Delta_g) \cap \sigma(\Delta) = \emptyset\}$ *est comaigre dans* $\mathcal{M}^1$*.*

Alors, nous obtenons immédiatement le corollaire suivant:

Corollaire 1.5.6. *Fixez* $\Delta = \frac{d^2}{dx^2}$*, pour toute métrique* $g \in \mathcal{G}_{uc}$*, le système* (1.5.13) *a une solution unique* $u_1 = u_2 = 0$*.*

En dimension 2, nous avons le résultat similaire:

Proposition 1.5.7. *En dimension* 2*, supposons que nous fixions une métrique* g_0 *et le laplacien* Δ_{g_0} *avec son spectre* $\sigma(\Delta_{g_0})$*. Alors l'ensemble* $\mathcal{G}_{uc} = \{g \in \mathcal{M}^2 : \sigma(\Delta_g) \cap \sigma(\Delta_{g_0}) = \emptyset\}$ *est comaigre dans* $\mathcal{M}^2$*.*

Et pour les détails de la preuve, nous nous référons à la section 3.5.4.

1.5.2 Couplées via des termes d'ordre zéro «en cascade»

Dans cette section, nous considérons principalement le Laplacien à coefficients constants. Il s'agit d'un travail conjoint avec Pierre Lissy. Dans cet article, nous avons prouvé la contrôlabilité d'un système des ondes couplées avec un seul contrôle et différentes vitesses.

Un modèle simple

D'adord, nous présentons un exemple simple comme suit:

$$\begin{cases} (\partial_t^2 - \Delta)u_1 + u_2 &= 0 \quad \text{dans }]0,T[\times\Omega, \\ (\partial_t^2 - 2\Delta)u_2 + u_3 &= 0 \quad \text{dans }]0,T[\times\Omega, \\ (\partial_t^2 - 2\Delta)u_3 &= f\mathbf{1}_\omega \quad \text{dans }]0,T[\times\Omega, \end{cases} \tag{1.5.14}$$

avec conditions de Dirichlet, où f est une fonction L^2 supportée dans $]0,T[\times\omega$. Par rapport à (1.5.1), nous considérons une structure de couplage en cascade pour les solutions. Notamment, le contrôle f n'agit directement que sur u_3, qui lui-même agit sur u_2 tandis que u_1 est contrôlé par u_2.

Pour ce système, nous avons $(u_1, u_2, u_3) \in H^4 \times H^2 \times H^1$ avec des conditions initiales nulle. En effet, puisque u_3 satisfait une équation d'onde avec un terme source $f \in L^1(]0,T[, L^2)$, il est classique qu'il existe une solution unique $u_3 \in C^1([0,T], H_0^1) \cap C^0([0,T], L^2)$. Puisque u_2 satisfait une équation d'onde avec un terme source $-u_3$, alors $u_2 \in C^1([0,T],H^2) \cap C^0([0,T],H_0^1)$. Pour u_1, de même, on obtient que $u_1 \in C^1([0,T],H^3) \cap C^0([0,T],H^2)$. Maintenant, nous avons besoin d'énoncer une propriété de régularité supplémentaire pour u_1. En appliquant l'opérateur de d'Alembert $\Box_2 = \partial_t^2 - 2\Delta$ des deux côtés de l'équation de $\Box_1 u_1 = (\partial_t^2 - \Delta)u_1 = -u_2$, on obtient que

$$\Box_2 \Box_1 u_1 = -\Box_2 u_2.$$

Puisque $\Box_2 u_2 = -u_3$, on obtient alors que $\Box_2\Box_1 u_1 = u_3$. Nous considérons que $\Box_2 u_1$ satisfait une équation d'onde avec un terme source u_3. Par conséquent, nous savons que $\Box_2 u_1 \in C^1([0,T],H^2) \cap C^0([0,T],H_0^1)$. Puisque $\Box_1 u_1 = -u_2 \in C^1([0,T],H^2) \cap C^0([0,T],H_0^1)$, nous savons que $\Delta u_1 = \Box_1 u_1 - \Box_2 u_1 \in C^1([0,T],H^2) \cap C^0([0,T],H_0^1)$. Donc, nous savons que $u_1 \in C^1([0,T],H^4) \cap C^0([0,T],H^3)$. Alors, nous remarquons un résultat de régularité $(u_1,u_2,u_3) \in H^4 \times H^2 \times H^1$. On peut se référer à [20] pour une preuve différente.

De plus, avec des conditions initiales nulles, nous remarquons également qu'il existe une condition de compatibilité pour ce problème de contrôle, c'est-à-dire $(-\Delta)^2 u_1 + \Delta u_2 \in H_0^1$. En fait, faisons d'abord une reformulation pour le système.

$$\begin{cases} v_1 = D_t^3 u_1, \\ v_2 = D_t u_2, \\ v_3 = u_3. \end{cases} \tag{1.5.15}$$

Et (v_1, v_2, v_3) satisfait au système suivant:

$$\begin{cases} \Box_1 v_1 + D_t^2 v_2 = 0 \text{ dans }]0,T[\times\Omega, \\ \Box_2 v_2 + D_t v_3 = 0 \text{ dans }]0,T[\times\Omega, \\ \Box_2 v_3 = f \text{ dans }]0,T[\times\Omega. \end{cases} \tag{1.5.16}$$

Comme

$$-D_t^2 = 2\Box_1 - \Box_2, \tag{1.5.17}$$

on a

$$D_t^2 v_2 = -(2\Box_1 - \Box_2)v_2. \tag{1.5.18}$$

Donc,

$$\Box_1(v_1 - 2v_2) - D_t v_3 = 0. \tag{1.5.19}$$

On peut poser

$$y = D_t v_1 - 2D_t v_2. \tag{1.5.20}$$

Alors, on a une équation pour y

$$\Box_1(y + 2v_3) = f. \tag{1.5.21}$$

Pour y, en utilisant les équations, on a

$$\begin{aligned} y &= D_t v_1 - 2D_t v_2 \\ &= D_t^4 u_1 - 2D_t^2 u_2 \\ &= D_t^2(-\Delta u_1 + u_2 - 2u_2) \\ &= D_t^2(-\Delta u_1 - u_2) \\ &= (-\Delta)^2 u_1 + \Delta u_2 - u_3. \end{aligned}$$

Donc, on obtient

$$\Box_1((-\Delta)^2 u_1 + \Delta u_2 + u_3) = f.$$

De plus, on a $(-\Delta)^2 u_1 + \Delta u_2 + u_3 \in H_0^1$, c'est-à-dire, $(-\Delta)^2 u_1 + \Delta u_2 \in H_0^1$. En considérant la régularité de u_1 et u_2, nous savons que $(u_1, u_2) \in H^4 \times H^2$. Par conséquent, nous pouvons seulement obtenir $(-\Delta)^2 u_1 + \Delta u_2 \in L^2$. Nous devons considérer non seulement la régularité des solutions mais aussi les conditions de compatibilité associées à la structure de couplage. Ceci est très différent du système sans couplage, et même différent du système d'onde couplé par la même vitesse ou des systèmes paraboliques couplés. A notre connaissance, il s'agit d'une caractéristique unique pour ce type de systèmes d'ondes couplés. Cela nous motive à considérer un système plus général avec le même type de structure de couplage.

la contrôlabilité d'un système d'équations d'ondes à vitesses différentes

On considère le système suivant:

$$\begin{cases} (\partial_t^2 - D\Delta)U + AU & = \hat{b} f \mathbf{1}_\omega & \text{dans }]0,T[\times\Omega, \\ U & = 0 & \text{sur }]0,T[\times\partial\Omega, \\ (U, \partial_t U)|_{t=0} & = (U^0, U^1) & \text{dans } \Omega, \end{cases} \tag{1.5.22}$$

avec

$$D = \begin{pmatrix} d_1 Id_{n_1} & 0 \\ 0 & d_2 Id_{n_2} \end{pmatrix}_{n\times n}, A = \begin{pmatrix} 0 & A_1 \\ 0 & A_2 \end{pmatrix}_{n\times n}, \text{ and } \hat{b} = \begin{pmatrix} 0 \\ b \end{pmatrix}_{n\times 1}, \tag{1.5.23}$$

où $n = n_1 + n_2$ et $d_1 \neq d_2$. $A_1 \in \mathcal{M}_{n_1,n_2}(\mathbb{R})$ et $A_2 \in \mathcal{M}_{n_2}(\mathbb{R})$ sont deux matrices de couplage données et $b \in \mathbb{R}^{n_2}$.

Nous avons les propriétés importantes et cruciales du système (1.5.22) : tous les coefficients sont constants, le couplage est en structure de cascade (notamment, la commande f n'agit directement que sur U_2, qui elle-même agit sur U_1 par la matrice A_1), et nous nous limitons au cas d'une commande scalaire (*i.e.* $f \in L^2(]0,T[, \mathbb{R}^m)$ avec $m = 1$).

Dans la proposition suivante, nous donnons une condition équivalente de la condition de rang de l'opérateur Kalman associée au système (1.5.22), qui est très spécifique à notre structure de couplage particulière et au fait que nous avons un seul contrôle.

Proposition 1.5.8. *Nous désignons par $\mathcal{K} = [-D\Delta + A|\hat{B}]$ l'opérateur de Kalman associé au système* (1.5.22)*. Alors, $Ker(\mathcal{K}^*) = \{0\}$ est équivalent à la satisfaction de toutes les conditions suivantes*

1. *$n_1 = 1$;*

2. *(A_2, B) satisfait la condition algébrique de rang de Kalman (voir Définition 1.2.3);*

3. *Supposons que $A_1 = \alpha = (\alpha_1, \cdots, \alpha_{n_2})$. Alors $\forall \lambda \in \sigma(-\Delta)$, α satisfait*

$$\alpha \left(\sum_{k=0}^{n_2-2} (d_1 - d_2)^k \lambda^k \sum_{j=k+1}^{n_2} a_j A_2^{j-1-k} + (d_1 - d_2)^{n_2-1} \lambda^{n_2-1} Id_{n_2} \right) \widehat{b} \neq 0, \tag{1.5.24}$$

où $(a_j)_{0 \leq j \leq n_2}$ sont les coefficients du polynôme caractéristique de la matrice A_2, c'est-à-dire $\chi(X) = X^{n_2} + \sum_{j=0}^{n_2-1} a_j X^j$, avec la convention que $a_{n_2} = 1$.

Avec cette condition équivalente, nous pouvons simplifier le système:

$$\begin{cases} \Box_1 u_1^1 + \sum_{j=1}^{s} \alpha_s u_j^2 = 0 \text{ dans }]0, T[\times\Omega, \\ \Box_2 u_1^2 + u_2^2 = 0 \text{ dans }]0, T[\times\Omega, \\ \vdots \\ \Box_2 u_{n_2-1}^2 + u_{n_2}^2 = 0 \text{ dans }]0, T[\times\Omega, \\ \Box_2 u_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j} u_j^2 = f \mathbb{1}_\omega \text{ dans }]0, T[\times\Omega, \\ u_1^1 = 0, u_j^2 = 0 \quad \text{sur }]0, T[\times\partial\Omega, 1 \leq j \leq n_2, \\ (u_1^1, u_1^2, \cdots, u_{n_2}^2)|_{t=0} = (u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) \text{ dans } \Omega, \\ (\partial_t u_1^1, \partial_t u_1^2, \cdots, \partial_t u_{n_2}^2)|_{t=0} = (u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) \text{ dans } \Omega. \end{cases} \tag{1.5.25}$$

Ici $n_1 = 1$, $A_1 = (\alpha_1, \cdots, \alpha_s, 0, \cdots, 0)$ et

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Puisque nous considérons le problème de contrôle dans un domaine Ω avec frontière, il est naturel pour nous d'introduire les espaces de Hilbert suivants $H_\Omega^s(\Delta)$.

Définition 1.5.9. *Nous désignons par $(\beta_j^2)_{j \in \mathbb{N}^*}$ la séquence non décroissante de valeurs propres (positives) de l'opérateur de Laplace $-\Delta$ avec condition de Dirichlet, répétée avec multiplicité, et $(e_j)_{j \in \mathbb{N}^*}$ une base orthonormée de $L^2(\Omega)$ constituée de fonctions propres associées à $(\beta_j^2)_{j \in \mathbb{N}^*}$:*

$$-\Delta e_j = \beta_j^2 e_j, \quad ||e_j||_{L^2} = 1.$$

Pour tout $s \in \mathbb{R}$, nous désignons par $H^s_\Omega(\Delta)$ l'espace de Hilbert défini par

$$H^s_\Omega(\Delta) = \{u = \sum_{j\in\mathbb{N}^*} a_j e_j; \sum_{j\in\mathbb{N}^*} (1+\beta_j^2)^s |a_j|^2 < \infty\}. \tag{1.5.26}$$

Sous cette structure particulière de couplage, nous introduisons des conditions de compatibilité appropriées pour le système (1.5.25). Désignons par $\mathcal{H}_r$ l'espace suivant

$$\begin{aligned}\mathcal{H}_r = \{&(u, v_1, \cdots, v_{n_2}) \in H^{n_2-s+2+r}_\Omega(\Delta) \times H^{n_2-1+r}_\Omega(\Delta) \times \cdots \times H^r_\Omega(\Delta) \text{ t.q.}\\ &U^r_{comp} \in H^r_\Omega(\Delta_D)\},\end{aligned} \tag{1.5.27}$$

où

$$\begin{aligned}U^r_{comp} = \Big(&(-d_1\Delta)^{n_2-s+1} u\\ &+ \sum_{k=0}^{n_2-s} \sum_{j=1}^{s} \sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l} (-d_1\Delta)^k (-d_2\Delta)^{n_2-s-k-l} v_{j+l}\\ &+ \sum_{j=1}^{s} \sum_{k=0}^{n_2-2s+j} \sum_{l=0}^{n_2-s-k} \frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}} \binom{n_2-s-k}{l} (-d_2\Delta)^{n_2-s-k-l} v_{j+k+l}\Big).\end{aligned} \tag{1.5.28}$$

Définition 1.5.10. *L'espace d'état du système* (1.5.25) *est défini par*

$$\mathcal{H}_1 \times \mathcal{H}_0.$$

Les deux conditions

$$\begin{aligned}U^1_{comp}(u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) \in H^1_\Omega(\Delta_D),\\ U^0_{comp}(u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) \in H^0_\Omega(\Delta_D)\end{aligned}$$

sont appelées les conditions de compatibilité pour la contrôlabilité du système (1.5.25).

Avec ces espaces bien préparés, nous obtenons le résultat suivant :

Théorème 1.5.11. *Pour $T > 0$, supposons que:*

1. *(ω, T, p_{d_i}) satisfait CCG, $i = 1, 2$.*
2. *Ω n'a pas de contact d'ordre infini avec ses tangentes.*
3. *L'opérateur de Kalman $\mathcal{K} = [-D\Delta + A|\hat{B}]$ satisfait à la condition de rang de l'opérateur de Kalman, c'est-à-dire que $Ker(\mathcal{K}^*) = \{0\}$.*

Alors le système (1.5.22) est exactement contrôlable.

Nous prouvons le théorème ci-dessus en trois étapes.

1. Étape 1: Nous simplifions le système (1.5.22), en utilisant la forme normale de Brunovský. Ceci est basé sur la Proposition 1.5.8 et nous avons seulement besoin de prouver la contrôlabilité exacte pour le système simplifié.

2. Étape 2: Nous utilisons les schémas d'itération pour obtenir les conditions de compatibilité associées à la structure de couplage dans le système (1.5.22). Par conséquent, nous préparons les espaces d'état appropriés pour la contrôlabilité du système.

3. Étape 3: Nous utilisons la méthode d'unicité de Hilbert pour dériver l'inégalité d'observabilité, puis nous suivons la même procédure que dans la section précédente. Nous établissons une inégalité d'observabilité faible et prouvons cette inégalité d'observabilité faible par l'argument de contradiction et la propagation des mesures de défaut pour les systèmes. Enfin, la propriété de continuation unique est donnée par la condition de rang de Kalman.

Chapter 2

Introduction (English)

2.1 Motivations

The controllability of the wave equations is a classic research topic in both the control theory and the analysis of partial differential equations. There is a large literature on the controllability of linear wave equations. One of the best results on this subject has been obtained by Bardos, Lebeau and Rauch in their article [10], where they introduced the famous geometric control condition and presented the application of the microlocal analysis in the subject. We can also refer to the paper [14] by Burq and Gérard and the paper [12] by Burq for the improvements or a simpler proof. These results form a basic backgrounds and also provide the main strategy for us to study the controllability of the wave equations.

As we can see, for a single wave equation, the exact controllability is by now well-known. There is a large literature on the controllability of a scalar wave equation through different approaches such as [10] by using microlocal analysis as we mentioned before, [38, 29] by using multipliers, [25, 11] by using Carleman estimates, or a completely constructive proof [30], etc.

Although we now have a better picture on the controllabilty of a single wave equation, the controllability of systems of wave equations is still not totally understood. To our knowledge, most of the references concern the case of systems with the same principal symbol. Alabau-Boussouira and Léautaud [5] studied the indirect controllability of two coupled wave equations, in which their controllability result was established using a multi-level energy method introduced in [2], and also used in [3, 4]. Liard and Lissy [37], Lissy and Zuazua [40] studied the observability and controllability of the coupled wave systems under the Kalman type rank condition. Moreover, we can find other controllability results for coupled wave systems, for example, Cui, Laurent, and Wang [19] studied the observability of wave equations coupled by first or zero order terms on a compact manifold.

However, when we consider the controllability of the wave system coupled with different speeds, there are very few results.

On the other hand, considering the controllability of a parabolic system, we find that there are no differences between the coupling with same speed and different speeds (for instance, see [6]). This also motivates us to investigate the results on the controllability of the wave system with different speeds.

In this thesis, the main model under study is the wave equation in the following form. Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, be a bounded, and smooth domain. For positive constants α and β, let $k_{ij}(x) : \Omega \to \mathbb{R}$, $1 \leq i, j \leq d$ be smooth functions which satisfy:

$$k_{ij}(x) = k_{ji}(x), \alpha|\xi|^2 \leq \sum_{1\leq i,j\leq d} k_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2, \forall x \in \Omega, \forall \xi \in \mathbb{R}^d. \tag{2.1.1}$$

Define $K(x)$ to be the symmetric positive definite matrix of coefficients $k_{ij}(x)$. Moreover, we define the density function $\kappa(x) = \frac{1}{\sqrt{\det(K(x))}}$. We also define the Laplacian by $\Delta_K = \frac{1}{\kappa(x)} div(\kappa(x) K \nabla \cdot)$ on Ω and the d'Alembert operator $\Box_K = \partial_t^2 - \Delta_K$ on $\mathbb{R}_t \times \Omega$. we consider a nonhomogeneous wave equation with a source term f:

$$\Box_K u = f, \tag{2.1.2}$$

with initial conditions:

$$u|_{t=0} = u^0, \partial_t u|_{t=0} = u^1. \tag{2.1.3}$$

2.2 Preliminaries

In this section, we shall introduce some basic aspects in the control problem of wave equations. We assume that ω is a nonempty open subset of Ω. We consider the interior controllability problem for the following wave equation:

$$\begin{cases} \Box_K u = f\mathbf{1}_\omega \text{ in } (0,T) \times \Omega, \\ u = 0 \quad \text{ on } (0,T) \times \partial\Omega, \\ u|_{t=0} = u^0(x), \quad \partial_t u|_{t=0} = u^1(x), \end{cases} \tag{2.2.1}$$

where f is a control function with support only localized in the subdomain ω.

It is well known that the wave equation models many physical phenomena such as small vibrations of elastic bodies and the propagation of sound. For instance (2.2.1) provides a good approximation for the small amplitude vibrations of an elastic string or a flexible membrane occupying the region Ω at rest. The control f represents then a localized force acting on the vibrating structure.

In addition, since the wave equation is the most relevant hyperbolic equations. Through the study of the wave equation, it helps us to understand how the properties of the hyperbolic equations act on the control problems.

Therefore it is interesting and important to study the controllability of the wave equation as one of the fundamental models of continuum mechanics and, at the same time, as one of the most representative equations in the theory of control of partial differential equations.

2.2.1 Controllability

In this section, we shall introduce several different types of the controllability for the wave equation (2.2.1).

Definition 2.2.1 (Controllability)**.** *Let $T > 0$.*

1. *(Exact controllability) We say that the wave equation* (2.2.1) *is exactly controllable in $H_0^1 \times L^2$ in time T if for any initial data $(u^0, u^1) \in H_0^1 \times L^2$ and target data $(\tilde{u}^0, \tilde{u}^1) \in H_0^1 \times L^2$, there exists a control function $f \in L^2((0,T) \times \omega)$ such that the solution of* (2.2.1) *issued from $(u|_{t=0}, \partial_t u|_{t=0}) = (u^0, u^1)$, satisfies $(u|_{t=T}, \partial_t u|_{t=T}) = (\tilde{u}^0, \tilde{u}^1)$.*

2. *(Null controllability) We say that the wave equation* (2.2.1) *is null controllable in $H_0^1 \times L^2$ in time T if for any initial data $(u^0, u^1) \in H_0^1 \times L^2$, there exists a control function $f \in L^2((0,T) \times \omega)$ such that the solution of* (2.2.1) *issued from $(u|_{t=0}, \partial_t u|_{t=0}) = (u^0, u^1)$, satisfies $(u|_{t=T}, \partial_t u|_{t=T}) = (0, 0)$.*

3. *(Controllability from zero) We say that the wave equation* (2.2.1) *is controllable from zero in $H_0^1 \times L^2$ in time T if for target data $(\tilde{u}^0, \tilde{u}^1) \in H_0^1 \times L^2$, there exists a control function $f \in L^2((0,T) \times \omega)$ such that the solution of* (2.2.1) *issued from $(u|_{t=0}, \partial_t u|_{t=0}) = (0, 0)$, satisfies $(u|_{t=T}, \partial_t u|_{t=T}) = (\tilde{u}^0, \tilde{u}^1)$.*

4. *(Partial controllability) Let Π be a projection operator defined in $H_0^1 \times L^2$. We say that the wave equation* (2.2.1) *is $\Pi-$exactly controllable in $H_0^1 \times L^2$ in time T if for any initial data $(u^0, u^1) \in H_0^1 \times L^2$ and target data $(\tilde{u}^0, \tilde{u}^1) \in H_0^1 \times L^2$, there exists a control function $f \in L^2((0,T) \times \omega)$ such that the solution of* (2.2.1) *issued from $(u|_{t=0}, \partial_t u|_{t=0}) = (u^0, u^1)$, satisfies $\Pi(u|_{t=T}, \partial_t u|_{t=T}) = \Pi(\tilde{u}^0, \tilde{u}^1)$.*

Remark 2.2.2. *Since the wave equation we consider is linear and reversible in time, the exact controllability, null controllability and the controllability from zero are all equivalent (one can refer to [17, Theorem 2.41]).*

2.2.2 Kalman conditions

In this section, we recall some Kalman rank conditions introduced in the literature of coupled parabolic systems and the link between them. First of all, we recall the usual Kalman rank condition for the controllability of linear autonomous ordinary differential equations (see *e.g.* [28]).

Definition 2.2.3 (Usual algebraic Kalman rank condition). *Let m, n be two positive integers. Assume $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_{n,m}(\mathbb{R})$. We introduce the Kalman matrix associated to A and B given by $[A|B] = [A^{n-1}B|\cdots|AB|B] \in \mathcal{M}_{n,nm}(\mathbb{R})$. We say that (A, B) satisfies the Kalman rank condition if $[A|B]$ is of full rank.*

This Kalman's type conditions for controllability are introduced in [28], which is a criterion for the time invariant linear control system $\dot{x} = Ax + Bu$ with a control $u \in L^\infty(]T_0, T_1[, \mathbb{R}^m)$. Moreover, we notice that the Kalman rank condition is an equivalent condition for the controllability of the time invariant linear control system $\dot{x} = Ax + Bu$ (one can refer to [17, Remark 1.17]).

Definition 2.2.4 (Kalman operator). *Assume that $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times m}$. Moreover, let $D \in \mathbb{R}^{n\times n}$ be a diagonal matrix. Then, the Kalman operator associated with $(-D\Delta + X, Y)$ is the matrix operator $\mathscr{K} = [-D\Delta + X|Y] : D(\mathscr{K}) \subset (L^2)^{nm} \to (L^2)^n)$, where the domain of the Kalman operator $\mathcal{D}(\mathscr{K}) = \{u \in (L^2)^{nm} : \mathscr{K} u \in (L^2)^n\}$.*

Definition 2.2.5 (Operator Kalman rank condition). *We say that the Kalman operator $\mathscr{K}$ satisfies the operator Kalman rank condition if $Ker(\mathscr{K}^*) = \{0\}$.*

The operator Kalman rank condition can be reformulated as follows.

Proposition 2.2.6. *[6, Proposition 2.2] The operator Kalman rank condition is equivalent to the following spectral Kalman rank condition:*

$$rank[(\lambda D + X)|Y] = n, \forall \lambda \in \sigma(-\Delta).$$

In particular, let $C > 0$ be a constant and $D = CId_n$. Then, the operator Kalman rank condition is equivalent to the usual algebraic Kalman rank condition given in Definition 2.2.3 (see [6, Remark 1.2]).

2.2.3 Hilbert uniqueness method

For the wave equation (2.2.1), we introduce the adjoint equation as follows:

$$\begin{cases} \Box_K v = 0 \text{ in } (0,T)\times\Omega, \\ v = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ v|_{t=0} = v^0(x), \quad \partial_t v|_{t=0} = v^1(x), \end{cases} \tag{2.2.2}$$

Definition 2.2.7. *We say a homogeneous wave equation* (2.2.2) *is observable in* $[0,T]\times\omega$ *if there exists a constant* $C>0$ *such that every solution* $v\in C^0(0,T,L^2)\cap C^1(0,T,H^{-1})$ *of the homogeneous wave equation* (2.2.2) *satisfies*

$$C\int_0^T\int_\omega |\kappa v|^2 dxdt \geq ||v^0||_{L^2}^2 + ||v^1||_{H^{-1}}^2. \tag{2.2.3}$$

Here the inequality (2.2.3) is called the observability inequality for the adjoint equation (2.2.2).

According to the Hilbert Uniqueness Method of J.-L. Lions [38], the controllability property is equivalent to an observability inequality for the adjoint system.

Theorem 2.2.8. *The wave equation* (2.2.1) *is null controllable if and only if the adjoint equation* (2.2.2) *is observable in* $[0,T]\times\omega$.

The proof idea of this theorem is the so-called Hilbert uniqueness method (HUM), which establishes the duality between the null controllability and the obsevability. We define the operator R by

$$R : f \in L^2((0,T)\times\omega) \mapsto (u^0,u^1) \in H_0^1 \times L^2, \tag{2.2.4}$$

where u is the solution of (2.2.1) with $(u|_{t=T}, \partial_t u|_{t=T}) = (0,0)$. On the other hand, we define the operator S by

$$S : (v^0,v^1) \in L^2 \times H^{-1} \mapsto bv\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \in L^2((0,T)\times\omega), \tag{2.2.5}$$

where v solves the adjoint equation (2.2.2). Therefore, the null controllability is just the surjectivity of the operator R and the observability is just the coercivity of the operator S. The Theorem 2.2.8 implies the duality $R^* = S$.

2.2.4 Geometric control condition

In order to study the observability inequality, a classical method is to follow the abstract three-step process initialized by Rauch and Taylor [46](see also [10]). It can be detailed as follows:

- Firstly, get the microlocal information on the observable region. Argue by contradiction to obtain different kinds of convergence in subdomain $(0,T)\times\omega$ and the whole domain $(0,T)\times\Omega$.
- Secondly, use microlocal defect measure (which is due to Gérard [23] and Tartar [47]), or propagation of singulaties theorem (see [26, Section 18.1]) to prove a weak observability estimate:

$$||v^0||_{L^2}^2 + ||v^1||_{H^{-1}}^2 \leq C(\int_0^T\int_\omega |b\kappa v|^2 dxdt + ||v^0||_{H^{-1}}^2 + ||v^1||_{H^{-2}}^2).$$

- Thirdly, use unique continuation properties of eigenfunctions to obtain the original observability inequality Equation (2.2.3).

For the high frequency estimates, a very natural condition is to assume that the control set satisfies the Geometric Control Condition(GCC).

Definition 2.2.9. *For $\omega \subset \Omega$ and $T > 0$, we shall say that the pair (ω, T, p_K) satisfies GCC if every general bicharacteristic of p_K meets ω in a time $t < T$, where p_K is the principal symbol of $\Box_K$.*

We will give the definition of bicharacteristics in Subsection 2.3.1. This condition was raised by Bardos, Lebeau, and Rauch [9] when they considered the controllability of a scalar wave equation and has now become a basic assumption for the controllability of wave equations. In [14], the authors show that the geometric control condition is a necessary and sufficient condition for the exact controllability of the wave equation with Dirichlet boundary conditions and continuous boundary control functions.

2.2.5 Unique continuation properties

For the low frequencies of the observability inequality, this reduces to prove a unique continuation property of the eigenfunctions of the Laplacian. That is to say, if ϕ satisfies the equation

$$-\Delta_K \phi = \lambda \phi, \lambda \in \mathbb{C}, \tag{2.2.6}$$

and $\phi|_\omega = 0$, can we obtain that $\phi \equiv 0$ in Ω.

2.3 Microlocal defect measures for wave equations

2.3.1 Geometric Preliminaries

Let $B = \{y \in \mathbb{R}^d : |y| < 1\}$ be the unit ball in $\mathbb{R}^d$. In a tubular neighbourhood of the boundary, we can identify $M = \Omega \times \mathbb{R}_t$ locally as $[0, 1[\times B$. More precisely, for $z \in \overline{M} = \overline{\Omega} \times \mathbb{R}_t$, we note that $z = (x, y)$, where $x \in [0, 1[$ and $y \in B$ and $z \in \partial M = \partial\Omega \times \mathbb{R}_t$ if and only if $z = (0, y)$. Now we consider $R = R(x, y, D_y)$ which is a second order scalar, self-adjoint, classical, tangential and smooth pseudo-differential operator, defined in a neighbourhood of $[0, 1] \times B$ with a real principal symbol $r(x, y, \eta)$, such that

$$\frac{\partial r}{\partial \eta} \neq 0 \text{ for } (x, y) \in [0, 1[\times B \text{ and } \eta \neq 0. \tag{2.3.1}$$

Let $Q_0(x,y,D_y)$, $Q_1(x,y,D_y)$ be smooth classical tangential pseudo-differential operators defined in a neighbourhood of $[0,1]\times B$, of order 0 and 1, and principal symbols $q_0(x,y,\eta)$, $q_1(x,y,\eta)$, respectively. Denote $P=(\partial_x^2+R)Id+Q_0\partial_x+Q_1$. The principal symbol of P is

$$p=-\xi^2+r(x,y,\eta). \tag{2.3.2}$$

We use the usual notations TM and T^*M to denote the tangent bundle and cotangent bundle corresponding to M, with the canonical projection π

$$\pi : TM(\text{ or } T^*M)\to M.$$

Denote $r_0(y,\eta)=r(0,y,\eta)$. Then we can decompose $T^*\partial M$ into the disjoint union $\mathcal{E}\cup\mathcal{G}\cup\mathcal{H}$, where

$$\mathcal{E}=\{r_0<0\},\quad \mathcal{G}=\{r_0=0\},\quad \mathcal{H}=\{r_0>0\}. \tag{2.3.3}$$

The sets $\mathcal{E}$, $\mathcal{G}$, $\mathcal{H}$ are called elliptic, glancing, and hyperbolic set, respectively. Define $\mathrm{Char(P)}=\{(x,y,\xi,\eta)\in T^*\mathbb{R}^{d+1}|_{\overline{M}}:\xi^2=r(x,y,\xi,\eta)\}$ to be the characteristic manifold of P. For more details, see [15] and [13].

2.3.2 Generalised bicharacteristic flow

We begin with the definition of the Hamiltonian vector field. For a symplectic manifold S with local coordinates (z,ζ), a Hamiltonian vector field associated with a real valued smooth function f is defined by the expression:

$$H_f=\frac{\partial f}{\partial\zeta}\frac{\partial}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial}{\partial\zeta}.$$

Considering the principal symbol p, we can also consider the associated Hamiltonian vector field H_p. The integral curve of this Hamiltonian H_p, denoted by γ, is called a bicharacteristic of p. Our next goal is to study the behavior of the bicharacteristic near the boundary. To describe the different phenomena when a bicharacteristic approaches the boundary, we need a more accurate decomposition of the glancing set $\mathcal{G}$. Let $r_1=\partial_x r|_{x=0}$. Then we can define the decomposition $\mathcal{G}=\bigcup_{j=2}^{\infty}\mathcal{G}^j$, with

$$\begin{aligned}
\mathcal{G}^2&=\{(y,\eta):r_0(y,\eta)=0,r_1(y,\eta)\neq 0\},\\
\mathcal{G}^3&=\{(y,\eta):r_0(y,\eta)=0,r_1(y,\eta)=0,H_{r_0}(r_1)\neq 0\},\\
&\vdots\\
\mathcal{G}^{k+3}&=\{(y,\eta):r_0(y,\eta)=0,H_{r_0}^j(r_1)=0,\forall j\leq k,H_{r_0}^{k+1}(r_1)\neq 0\},\\
&\vdots\\
\mathcal{G}^{\infty}&=\{(y,\eta):r_0(y,\eta)=0,H_{r_0}^j(r_1)=0,\forall j\}.
\end{aligned}$$

Here $H_{r_0}^j$ is just the vector field H_{r_0} composed j times. Moreover, for $\mathcal{G}^2$, we can define $\mathcal{G}^{2,\pm} = \{(y,\eta) : r_0(y,\eta) = 0, \pm r_1(y,\eta) > 0\}$. Thus $\mathcal{G}^2 = \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}$. For $\rho \in \mathcal{G}^{2,+}$, we say that ρ is a gliding point and for $\rho \in \mathcal{G}^{2,-}$, we say that ρ is a diffractive point. For $\rho \in \mathcal{G}^j$, $j \geq 2$, we say that a bicharacterisric of p tangentially contact the boundary $\{x = 0\} \times B$ with order j at the point ρ.

Consider a bicharacteristic $\gamma(s)$ with $\pi(\gamma(0)) \in M$ and $\pi(\gamma(s_0)) \in \partial M$ be the first point which touches the boundary. Then if $\gamma(s_0) \in \mathcal{H}$, we can define $\xi^{\pm}(\gamma(s_0)) = \pm\sqrt{r_0(\gamma(s_0))}$, which are the two different roots of $\xi^2 = r_0$ at the point $\gamma(s_0)$. Notice that the bicharacteristic with the direction ξ^- will leave the domain M while the bicharacteristic with the other direction ξ^+ will enter into the interior of M. This leads to a definition of the broken bicharacteristics(See [26] Section 24.2 for more details):

Definition 2.3.1. *A broken bicharacteristic of p is a map:*

$$s \in I \backslash D \mapsto \gamma(s) \in T^*M \backslash \{0\}$$

where I is an interval on $\mathbb{R}$ and D is a discrete subset, such that

1. *If J is an interval contained in $I \backslash D$, then for $s \in J \mapsto \gamma(s)$ is a bicharacteristic of p in M.*
2. *If $s \in D$, then the limits $\gamma(s^+)$ and $\gamma(s^-)$ exist and belongs to $T_z^*M \backslash \{0\}$ for some $z \in \partial M$, and the projections in $T_z^*\partial M \backslash \{0\}$ are the same hyperbolic point.*

If $\gamma(s_0) \in \mathcal{G}$, we have different situations. If $\gamma(s_0) \in \mathcal{G}^{2,+}$, then $\gamma(s)$, locally near s_0, passes transversally and enters into T^*M immediately. If $\gamma(s_0) \in \mathcal{G}^{2,-}$ or $\gamma(s_0) \in \mathcal{G}^k$ for some $k \geq 3$, then $\gamma(s)$ will continue inside $T^*\partial M$ and follow the Hamiltonian flow of H_{-r_0}. To be more precise, we have the definition of the generalized bicharacteristics(See [26] Section 24.3 for more details):

Definition 2.3.2. *A generalized bicharacteristic of p is a map:*

$$s \in I \backslash D \mapsto \gamma(s) \in T^*M \cup \mathcal{G}$$

where I is an interval on $\mathbb{R}$ and D is a discrete subset I such that $p \circ \gamma = 0$ and the following properties hold:

1. *$\gamma(s)$ is differentiable and $\frac{\mathrm{d}\gamma}{\mathrm{d}s} = H_p(\gamma(s))$ if $\gamma(s) \in T^*M$ or $\gamma(s) \in \mathcal{G}^{2,+}$.*
2. *Every $t \in D$ is isolated i.e. there exists $\epsilon > 0$ such that $\gamma(s) \in T^*\overline{M} \backslash T^*\partial M$ if $0 < |s - t| < \epsilon$, and the limits $\gamma(s^{\pm})$ are different points in the same hyperbolic fiber of $T^*\partial M$.*

3. $\gamma(s)$ *is differentiable and* $\frac{\mathrm{d}\gamma}{\mathrm{d}s} = H_{-r_0}(\gamma(s))$ *if* $\gamma(s) \in \mathcal{G} \backslash \mathcal{G}^{2,+}$.

Remark 2.3.3. *We denote the Melrose cotangent compressed bundle by* ${}^bT^*\overline{M}$ *and the associated canonical map by* $j : T^*\overline{M} \mapsto {}^bT^*\overline{M}$. j *is defined by*

$$j(x, y, \xi, \eta) = (x, y, x\xi, \eta).$$

Under this map j*, one could see* $\gamma(s)$ *as a continuous flow on the compressed cotangent bundle* ${}^bT^*\overline{M}$*. This is the so-called Melrose-Sjöstrand flow.*

From now on we always assume that there is no infinite tangential contact between the bicharacteristic of p and the boundary. This is in the meaning of the following definition:

Definition 2.3.4. *We say that there is no infinite contact between the bicharacteristics of* p *and the boundary if there exists* $N \in \mathbb{N}$ *such that the gliding set* $\mathcal{G}$ *satisfies*

$$\mathcal{G} = \bigcup_{j=2}^{N} \mathcal{G}^j.$$

It is well-known that under this hypothesis there exists a unique generalized bicharacteristic passing through any point. This means that the Melrose-Sjöstrand flow is globally well-defined. One can refer to [42] and [43] for the proof.

2.3.3 Microlocal defect measure

In this section, we will give two approaches to construct the microlocal defect measures. The first one is based on the article by Gérard and Leichtnam [24] for Helmoltz equation and Burq [13] for wave equations. The other one follows the idea in the article [31] by Lebeau and we rely on the article [15] by Burq and Lebeau for the setting of wave systems. In the first approach, we can compare two different measures, especially the supports of two different measures. Let $(u^k)_{k\in\mathbb{N}}$ be a bounded sequence in $\left(L^2_{loc}(\mathbb{R}^+; L^2(\Omega))\right)^n$, converging weakly to 0 and such that

$$\begin{cases} Pu^k = o(1)_{H^{-1}}, \\ u^k|_{\partial M} = 0. \end{cases} \tag{2.3.4}$$

Let $\underline{u}_k$ be the extension by 0 across the boundary of Ω. Then the sequence $\underline{u}_k$ is bounded in $\left(L^2_{loc}(\mathbb{R}_t; L^2(\mathbb{R}^d))\right)^n$. Let $\underline{\mathcal{A}}$ be the space of $n \times n$ matrices of classical polyhomogeneous pseudo-differential operators of order 0 with compact support in $\mathbb{R}^+ \times \mathbb{R}^d$ (i.e, $A = \varphi A \varphi$ for some $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$). Let us denote by $\underline{\mathcal{M}}^+$ the set of nonnegative Radon measures on $T^*(\mathbb{R}^+ \times \mathbb{R}^d)$. Following [13, Section 1], we have the existence of the microlocal defect measure as follows:

Proposition 2.3.5 (Existence of the microlocal defect measure-1)**.** *There exists a subsequence of $(\underline{u}^k)$ (still noted by $(\underline{u}^k)$) and $\underline{\mu} \in \underline{\mathcal{M}}^+$ such that*

$$\forall A \in \underline{\mathcal{A}}, \quad \lim_{k\to\infty} (A\underline{u}^k, \underline{u}^k)_{L^2} = \langle \underline{\mu}, \sigma(A) \rangle, \tag{2.3.5}$$

where $\sigma(A)$ is the principal symbol of the operator A (which is a matrix of smooth functions, homogeneous of order 0 in the variable ξ, i.e. a function on $S^*((\mathbb{R}^+ \times \mathbb{R}^d))$.

From [13, Théorème 15], we have the following proposition.

Proposition 2.3.6. *For the microlocal defect measure $\underline{\mu}$ defined above, we have the following properties.*

- *The measure $\underline{\mu}$ is supported on the intersection of the characteristic manifold with $\mathbb{R}^+ \times \overline{\Omega}$:*

$$supp(\underline{\mu}) \subset \mathrm{Char(P)} = \{(t, x, \tau, \xi); x \in \overline{M}, \tau^2 = |\xi|_x^2\}. \tag{2.3.6}$$

- *The measure $\underline{\mu}$ does not charge the hyperbolic points in ∂M :*

$$\underline{\mu}(\mathcal{H}) = 0.$$

- *In particular, if $n = 1$, the scalar measure $\underline{\mu}$ is invariant along the generalized bicharacteristic flow.*

Remark 2.3.7. *Notice first that in [13, Section 3], the author considered the case of solutions to the wave equation at the energy level (bounded in H^1_{loc}, and hence was considering second order operators. However, it is easy to pass from H^1 to L^2 solutions by applying the operator ∂_t and conversely from L^2 to H^1 by applying the operator ∂_t^{-1}, i.e. if v is an L^2 solution, considering the solution u associated to $\big((-\Delta_D)^{-1}(\partial_t v \mid_{t=0}), v \mid_{t=0}\big)$, which of course satisfies $\partial_t u = v$. This procedure amounts to replacing the test operators of order 0 A by the test operator of order 2, $B = -\partial_t \circ A \circ \partial_t$, but since τ^2 does not vanish on the characteristic manifold, it is an elliptic factor which changes nothing.*

Remark 2.3.8. *Notice also that due to discontinuity of the generalised bicharacteristics when they reflect on the boundary at hyperbolic points (the points corresponding to the left and right limits at $s \in D$), in Definition 2.3.1, the generalised bicharacteristic flow is not well defined (there are two points above any points corresponding to $s \in D$). However, since the measure μ does not charge these hyperbolic points, this flow is well defined μ almost surely and the invariance property makes sense. Notice also that in [13, Appendice], weaker property than invariance* *(namely that the support is a union of generalised bicahracteristics) is proved. The general result follows from this weaker result by applying the strategy in [31]. In any case, for the purpose of the present article, the invariance of the support would suffice.*

On the other hand, let $\mathcal{A}$ be the space of $n \times n$ matrices of pseudo-differential operators of order 0, in the form of $A = A_i + A_t$ with A_i classical pseudo-differential operator with compact support in M(i.e, $A_i = \varphi A_i \varphi$ for some $\varphi \in C_0^\infty(M)$) and A_t a classical tangential pseudo-differential operator in $\overline{M}$(i.e, $A_t = \varphi A_t \varphi$ for some $\varphi \in C^\infty(\overline{M})$). Then denote

$$Z = j(\mathrm{Char(P)}), \quad \hat{Z} = Z \cup j(T^*\overline{M}|_{x=0}),$$

where j is defined in (4.2.14) and

$$S\hat{Z} = (\hat{Z} \backslash \overline{M})/\mathbb{R}_+^*, \quad SZ = (Z \backslash \overline{M})/\mathbb{R}_+^*.$$

Remark 2.3.9. *$S\hat{Z}$ and SZ are the quotient spherical spaces of $\hat{Z}$ and Z and they are locally compact metric spaces.*

For $A \in \mathcal{A}$, with principal symbol $a = \sigma(A)$, define

$$\kappa(a)(\rho) = a(j^{-1}(\rho)), \forall \rho \in {}^bT^*\overline{M}.$$

Now, we have that $\mathcal{K} = \{\kappa(a) : a = \sigma(A), A \in \mathcal{A}\} \subset C^0(S\hat{Z}; End(\mathbb{C}^n))$. Define $\mathcal{M}^+$ to be the space of all positive Borel measures on $S\hat{Z}$. By duality, we know that $\mathcal{M}^+$ is the dual space of $C_0^0(S\hat{Z}; End(\mathbb{C}^n))$, which verifies the property:

$$\langle \mu, a \rangle \geq 0, \forall a \in C^0(S\hat{Z}; End^+(\mathbb{C}^n)), \forall \mu \in \mathcal{M}^+,$$

where $End^+(\mathbb{C}^n)$ denotes the space of $n \times n$ positive hermitian matrices. Following the article [15] by Burq and Lebeau, we obtain the existence of the microlocal defect measure and some properties as follows:

Proposition 2.3.10 (Existence of the microlocal defect measure-2)**.** *There exists a subsequence of (u^k) (still noted by (u^k)) and $\mu \in \mathcal{M}^+$ such that*

$$\forall A \in \mathcal{A}, \quad \lim_{k \to \infty} (Au^k, u^k)_{L^2} = \langle \mu, \kappa(\sigma(A)) \rangle. \tag{2.3.7}$$

Lemma 2.3.11. *The microlocal defect measure μ defined in Proposition 2.3.10 satisfies that $\mu \mathbb{1}_{\mathcal{H} \cup \mathcal{E}} = 0$ where $\mathcal{H}$ is the set of hyperbolic points and $\mathcal{E}$ is the set of elliptic points as defined in Subsection 2.3.1.*

Remark 2.3.12. *From Proposition* 2.3.6*, we know that supp*($\underline{\mu}$) $\subset$ Char(P)*. Notice that in the interior of* M*, the two definitions coincide, i.e.,* $\underline{\mu}|_{\mathrm{Char(P)}} = \mu$ *in the interior of* M*. At the boundary, since both measures* $\underline{\mu}$ *and* μ *do not not charge the hyperbolic points in* ∂M*, we know that* $\underline{\mu}|_{S\hat{Z}} = \mu$ *holds* μ *almost surely and* $\underline{\mu}$ *almost surely. Under this sense, we can identify the two measures.*

In the following, suppose that there is no infinite contact between the bicharacteristic of p and the boundary. This hypothesis implies the existence and uniqueness of the generalized bicharacteristic passing through any point, which ensures that the Melrose-Sjöstrand flow is globally well-defined. By a suitable change of parameter along this flow, we obtain a flow on SZ. Consider S a hypersurface tranverse to the flow. Then locally, $SZ = \mathbb{R}_s \times S$ where s is the well-chosen parameter along the flow. We have the following propagation lemma for the microlocal defect measure.

Lemma 2.3.13. *Assume that the microlocal defect measure* μ *is defined in Proposition* 4.2.8*. Then* μ *is supported in* SZ *and there exists a function*

$$(s, z) \in \mathbb{R}_s \times S \mapsto M(s, z) \in \mathbb{C}^n$$

$\mu-$*almost everywhere continuous such that the pull back of the measure* μ *by* M*(i.e., the measure* $\mathcal{P}^*\mu = M^*\mu M$ *defined for* $a \in C^0(SZ)$*) by*

$$\langle M^*\mu M, a\rangle = \langle \mu, MaM^*\rangle$$

satisfies

$$\frac{d}{ds}\mathcal{P}^*\mu = 0.$$

We say that the measure μ *is invariant along the flow associated to* M*. Furthermore, the function* M *is continuous and along any generalized bicharacteristic the matrix* M *is solution to a differential equation whose coefficients can be explicitly computed in terms of the geometry and the different terms in the operator* P*.*

For the differential equation which M satisfies, one can refer to [15, Section 3.2] for more details.

Remark 2.3.14. *For a scalar wave equation, we know that the defect measure is invariant along the general bicharacteristic flow.*

Remark 2.3.15. *Roughly speaking, in the result above, the norm of* M *describes the damping of the measure* μ*, whereas the rotation component of* M *describes the way the polarization of the measure (asymptotic polarization of the sequence* (u^k)*) is turning.*

2.4 The controllability of a scalar wave equation

In this section, we provide a sketch proof for the controllability of a scalar wave equation as we introduced in (2.2.1):

$$\begin{cases} \Box_K u = f\mathbf{1}_\omega \text{ in } (0,T)\times\Omega, \\ u = 0 \quad \text{ on } (0,T)\times\partial\Omega, \\ u|_{t=0} = u^0(x), \quad \partial_t u|_{t=0} = u^1(x), \end{cases} \tag{2.4.1}$$

where we assume that $f \in L^2((0,T)\times\omega)$ and the initial data $(u^0,u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. We consider the null controllability of this equation. The proof is based on three steps as follows:

1. (HUM and observability) Applying the Hilbert uniqueness method, the controllability property is equivalent to an observability inequality for the adjoint system. To be more precise here, we only need to prove: $\exists C > 0$ such that for any solutions of the adjoint equation:

$$\begin{cases} \Box_K v = 0 \text{ in } (0,T)\times\Omega, \\ v = 0 \quad \text{ on } (0,T)\times\partial\Omega, \\ v|_{t=0} = v^0(x), \quad \partial_t v|_{t=0} = v^1(x), \end{cases} \tag{2.4.2}$$

   we have

$$||v^0||_{L^2}^2 + ||v^1||_{H^{-1}}^2 \leq C\int_0^T\int_\omega |v|^2 dxdt. \tag{2.4.3}$$

2. (High-frequency estimates) We first establish a weak observability inequality as follows:

$$||v^0||_{L^2}^2 + ||v^1||_{H^{-1}}^2 \leq C\left(\int_0^T\int_\omega |v|^2 dxdt + ||v^0||_{H^{-1}}^2 + ||v^1||_{H^{-2}}^2\right). \tag{2.4.4}$$

   We prove this inequality by the argument of contradiction. Suppose the inequality (2.4.4) is false, there exists a sequence $(v^{k,0}, v^{k,1})_{k\in\mathbb{N}}$ in $L^2 \times H^{-1}$ such that

$$||v^{k,0}||_{L^2}^2 + ||v^{k,1}||_{H^{-1}}^2 = 1, \tag{2.4.5}$$

$$||v^{k,0}||_{H^{-1}}^2 + ||v^{k,1}||_{H^{-2}}^2 \to 0, k\to\infty \tag{2.4.6}$$

$$\int_0^T\int_\omega |v^k|^2 dxdt \to 0, k\to\infty \tag{2.4.7}$$

   where v^k is the solution of (2.4.2) with initial data $(v^{k,0}, v^{k,1})$. Hence, there exists a microlocal defect measure μ associated with the bounded sequence

v^k. According to the previous section, we know that μ is invariant along the general bichacteristic flow. In addition, we know that $\mu|_{(0,T)\times\omega} = 0$ by (2.4.7). Hence, we obtain $\mu \equiv 0$. Combining with the energy conservation law of the homogenerous wave equation (2.4.2), there is a controdiction with the hypothesis (2.4.5). Therefore, we prove the weak observability inequality (2.4.4).

3. (Low-frequency estimates) We use the weak observability inequality (2.4.4) to prove the original observability (2.4.3). We also argue by contradiction. Suppose that (2.4.3) is false, then, there exists a sequence $(v^{k,0}, v^{k,1})_{k\in\mathbb{N}}$ in $L^2 \times H^{-1}$ such that

$$||v^{k,0}||^2_{L^2} + ||v^{k,1}||^2_{H^{-1}} = 1, \tag{2.4.8}$$

$$\int_0^T \int_\omega |v^k|^2 dxdt \to 0, k \to \infty \tag{2.4.9}$$

where v^k is the solution of (2.4.2) with initial data $(v^{k,0}, v^{k,1})$. Since we proved the weak observability inequality, we know that

$$1 = ||v^{k,0}||^2_{L^2} + ||v^{k,1}||^2_{H^{-1}} \leq C \left(\int_0^T \int_\omega |v^k|^2 dxdt + ||v^{k,0}||^2_{H^{-1}} + ||v^{k,1}||^2_{H^{-2}} \right). \tag{2.4.10}$$

Let (v^0, v^1) be the weak limit of $(v^{k,0}, v^{k,1})$, i.e.$(v^{k,0}, v^{k,1}) \rightharpoonup (v^0, v^1)$ in $L^2 \times H^{-1}$ and v be the solution of the adjoint equation (2.4.2) with initial data (v^0, v^1). Since $L^2 \times H^{-1} \mapsto H^{-1} \times H^{-2}$ is compact, we know that $||v^{k,0}||^2_{H^{-1}} + ||v^{k,1}||^2_{H^{-2}} \to ||v^0||^2_{H^{-1}} + ||v^1||^2_{H^{-2}}$. As a consequence, let k tends to infinity, we obtain that

$$1 \leq C \left(||v^0||^2_{H^{-1}} + ||v^1||^2_{H^{-2}} \right). \tag{2.4.11}$$

Then we analyze the space of the invisible solutions defined by

$$\mathcal{N}(T) = \{(w^0, w^1) \in L^2 \times H^{-1} : w(t,x) = 0, \text{ for } t \in (0,T), x \in \omega\}. \tag{2.4.12}$$

Here w is a solution of the adjoint equation (2.4.2) with initial data (w^0, w^1). Hence, $(v^0, v^1) \in \mathcal{N}(T)$. Next, we prove that $\mathcal{N}(T) = \{0\}$. According to (2.4.4), we know that $\mathcal{N}(T)$ has finite dimension. Define $\mathscr{A} = \begin{pmatrix} 0 & 1 \\ -\Delta_K & 0 \end{pmatrix}$. Then $\mathcal{N}(T)$ is stable under the application of $\mathscr{A}$. Therefore, $\mathcal{N}(T)$ contains an eigenvector of $\mathscr{A}$, i.e. $\exists \lambda \in \mathbb{C}$ and $(\phi_0, \phi_1) \in H^1_0 \times L^2$ such that

$$\begin{cases} \mathscr{A} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \lambda \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}, \text{ in } \Omega, \\ \phi_0 = 0, \text{ in } \omega. \end{cases} \tag{2.4.13}$$

This is equivalent to: for $\lambda \in \mathbb{C}$ and $\phi_0 \in H_0^1$

$$\begin{cases} -\Delta\phi_0 = \lambda^2\phi_0, \text{ in } \Omega, \\ \phi_0 = 0, \text{ in } \omega. \end{cases} \tag{2.4.14}$$

This is a classic unique continuation problem. Using Carleman estimates (see [16]), we obtain that $\phi_0 \equiv 0$. Consequently, we know that $\mathcal{N}(T) = \{0\}$. Therefore, we have $(v^0, v^1) = (0, 0)$, which is a contradiction with the hypothesis (2.4.11). Hence, we prove the observability inequality (2.4.3).

In summary, we first apply Hilbert uniqueness method to obtain the observability inequality. Then for high-frequency regime, we prove a weak observability inequality by the microlocal analysis. At last, for low-frequency regime, it is equivalent to proving a unique continuation property for some eigenfucntions. This is the basic strategy for us to deal with the controllability of the wave equations.

2.5 Coupled wave systems

2.5.1 Coupled by the control function

In this section, we consider the interior simultaneous controllability problem of a wave system with different speeds. One could find this result in my article [44].

A simple model

First we introduce a simple example as follows:

$$\begin{cases} (\partial_t^2 - \Delta)u_1 = f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \\ (\partial_t^2 - 2\Delta)u_2 = f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \\ u_j = 0 \quad \text{ on } (0,T) \times \partial\Omega, j = 1, 2, \\ u_j(0,x) = u_j^0(x) \in H_0^1, \quad \partial_t u_j(0,x) = u_j^1(x) \in L^2, j = 1, 2. \end{cases} \tag{2.5.1}$$

Notice that these two wave equations are of different speeds and we use the same control function $f \in L^2((0,T) \times \omega)$ to control both equations at the same time.

For our example (2.5.1), applying Hilbert uniqueness method, we only need to prove an observability inequality

$$\sum_{i=1}^{2}(||v_i^0||_{L^2}^2 + ||v_i^1||_{H^{-1}}^2) \leq C \int_0^T \int_\omega |v_1 + v_2|^2 dxdt \tag{2.5.2}$$

for solutions (v_1, v_2) of the adjoint system with initial data (v_i^0, v_i^1):

$$\begin{cases} (\partial_t^2 - \Delta)v_1 = 0 \\ (\partial_t^2 - 2\Delta)v_2 = 0 \end{cases} \tag{2.5.3}$$

To prove the inequality (2.5.2), we first look at the high-frequency regime. Since the two wave equations are of different speeds, then characteristic manifolds are disjoint, which implies that $||v_1 + v_2||^2_{L^2} \approx ||v_1||^2_{L^2} + ||v_2||^2_{L^2}$ in the high-frequency regime. With the application of the microlocal defect measures, we know that for high frequencies, observe the sum $v_1 + v_2$ is almost equivalent to observing each of them. Then, we look at the low-frequency regime. It is equivalent to considering a unique continuation problem for eigenfunctions as follows: only zero solutions satisfy that

$$\begin{cases} -\Delta\phi_1 = \lambda\phi_1 \text{ in } \Omega, \\ -2\Delta\phi_2 = \lambda\phi_2 \text{ in } \Omega, \\ \phi_1 + \phi_2 = 0 \text{ in } \omega. \end{cases} \tag{2.5.4}$$

In this example, this property is easy to prove. Since the eigenfunctions of the laplacian are analytic, we know that $\phi_1 + \phi_2 \equiv 0$ in the whole domain Ω. Then, by adding two equations together, we obtain that $\Delta\phi_2 = 0$. Combining with the Dirichlet boundary condition, we know that $\phi_2 \equiv 0$, which implies that $\phi_1 = -\phi_2 \equiv 0$. Hence, we are able to prove this simultaneous control problem. Therefore, we conclude three features of this kind of problem:

1. Wave equations are of different speeds while we use the same control function to control all these equations at the same time.

2. Considering the observability inequality, we use the localized norm (restricted in subdomain ω) of the sum of solutions to control the full energy norm of the initial data.

3. We need a unique continuation property for the eigenfunctions associated with the wave system.

This motivates us to consider the generalisation of this example.

Simultaneous control of wave systems

In my article [44], we consider the exact controllability on an open domain Ω of wave systems with space varying and different speeds coupled by a single control function acting on a open subset ω. To be more precise, we consider the simultaneous interior controlllability for the following wave system:

$$\begin{cases} \Box_{K_1} u_1 = b_1 f \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T)\times\Omega, \\ \Box_{K_2} u_2 = b_2 f \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T)\times\Omega, \\ \vdots \\ \Box_{K_n} u_n = b_n f \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T)\times\Omega, \\ u_j = 0 \quad \text{on } (0,T)\times\partial\Omega, 1 \le j \le n, \\ u_j(0,x) = u_j^0(x), \quad \partial_t u_j(0,x) = u_j^1(x), 1 \le j \le n. \end{cases} \tag{2.5.5}$$

Here, we choose $K_i (1 \leq i \leq n)$ to be n different symmetric positive definite matrices, which is a generalization of n different wave speeds of different constant metrics. In addition, it is also important that we apply the same control function f on each equation. b_i are n nonzero constant coefficients. We could see this example as a special case where the coupling only appears in the control function. For this system, we are able to prove the partial controllability result as follows:

Theorem 2.5.1. *Given $T > 0$, suppose that:*

1. (ω, T, p_{K_i}) *satisfies GCC,* $i = 1, 2, \cdots, n$,
2. $K_1 > K_2 > \cdots > K_n$ *in* ω,
3. Ω *has no infinite order of tangential contact on the boundary.*

Then, there exists a finite dimensional subspace $E \subset (H_0^1(\Omega) \times L^2(\Omega))^n$ such that the system (2.5.5) *is $\mathbb{P}-$exactly controllable, where $\mathbb{P}$ is the orthogonal projector on $E^{\perp}$.*

As we have presented before, in order to study the low frequencies, we need to introduce the notion of unique continuation of eigenfunctions.

Definition 2.5.2. *We say the system Equation* (3.1.2) *satisfies the unique continuation of eigenfunctions if the following property holds: $\forall \lambda \in \mathbb{C}$, the only solution $(\phi_1, \cdots, \phi_n) \in (H_0^1(\Omega))^n$ of*

$$\begin{cases} -\Delta_{K_1} \phi_1 = \lambda^2 \phi_1 \text{ in } \Omega, \\ -\Delta_{K_2} \phi_2 = \lambda^2 \phi_2 \text{ in } \Omega, \\ \cdots \\ -\Delta_{K_n} \phi_n = \lambda^2 \phi_n \text{ in } \Omega, \\ b_1 \kappa_1 \phi_1 + \cdots + b_n \kappa_n \phi_n = 0 \text{ in } \omega, \end{cases}$$

is the zero solution $(\phi_1, \cdots, \phi_n) \equiv 0$.

Remark 2.5.3. *As we present in the section 3.5.4, the unique continuation property does not hold true in some cases.*

Hence, we are able to obtain the exact/null controllability as follows:

Theorem 2.5.4. *Given $T > 0$, suppose that:*

1. (ω, T, p_{K_i}) *satisfies GCC,* $i = 1, 2, \cdots, n$,
2. $K_1 > K_2 > \cdots > K_n$ *in* ω,
3. Ω *has no infinite order of tangential contact on the boundary,*

4. *The system* (2.5.5) *satisfies the unique continuation property of eigenfunctions.*

Then the system (2.5.5) *is exactly controllable in* $(H_0^1(\Omega) \times L^2(\Omega))^n$.

As we present in the previous section, we prove this theorem by similar procedure. First, we apply the Hilbert Uniqueness Method, and obtian the observability inequality: $\exists C > 0$ such that for any solution of the adjoint system:

$$\begin{cases} \Box_{K_1} v_1 = 0 \text{ in } (0,T) \times \Omega, \\ \Box_{K_2} v_2 = 0 \text{ in } (0,T) \times \Omega, \\ \vdots \\ \Box_{K_n} v_n = 0 \text{ in } (0,T) \times \Omega, \\ v_j = 0 \quad \text{ on } (0,T) \times \partial\Omega, 1 \le j \le n, \\ (v_1(0,x), \partial_t v_1(0,x), \cdots, v_n(0,x)\partial_t v_n(0,x)) = V^0, \end{cases} \tag{2.5.6}$$

where $V^0 \in (L^2 \times H^{-1})^n$, we have

$$C \int_0^T \int_\omega |b_1 \kappa_1 v_1 + \cdots + b_n \kappa_n v_n|^2 dx dt \ge ||V^0||^2_{(L^2 \times H^{-1})^n}. \tag{2.5.7}$$

Then we only need to prove this observability inequality (2.5.7). Looking at the high-frequency, we prove a weak observability estimate:

$$||V^0||^2_{(L^2 \times H^{-1})^n} \le C \left(\int_0^T \int_\omega |\sum_{j=1}^n b_j \kappa_j v_j|^2 dx dt + ||V^0||^2_{(H^{-1} \times H^{-2})^n} \right). \tag{2.5.8}$$

Using the argument by contradiction, we assume that the above inequality was false, we could obtain a sequence $(V^{0,k})_{k \in \mathbb{N}}$ such that

$$||V^{0,k}||^2_{(L^2 \times H^{-1})^n} = 1, \tag{2.5.9}$$

$$\int_0^T \int_\omega |b_1 \kappa_1 v_1^k + \cdots + b_n \kappa_n v_n^k|^2 dx dt \to 0, k \to \infty, \tag{2.5.10}$$

and

$$||V^{0,k}||^2_{(H^{-1} \times H^{-2})^n} \to 0, k \to \infty. \tag{2.5.11}$$

Here we use $v_i^k (1 \le i \le n)$ to denote the corresponding solution of the system Equation (2.5.6) with the initial data $V^{0,k}$. Since we have the assumption 2, we know that the characteristic manifolds of each wave equation are disjoint, which implies that

$$\int_0^T \int_\omega |b_1 \kappa_1 v_1^k + \cdots + b_n \kappa_n v_n^k|^2 dx dt \approx \sum_{i=1}^n \int_0^T \int_\omega |b_i \kappa_i v_i^k|^2 dx dt \tag{2.5.12}$$

Hence, we know that each defect measure μ_i associated with v_i^k is zero through the application of the propagation of the defect measures and the Geometric control condition. This provides a contradiction with the normalized norm of initial data, i.e. $||V^{0,k}||^2_{(L^2\times H^{-1})^n} = 1$. Then we combine the assumption (4), we know that the observability inequality is true. This gives us the result of the exact/null controllability of the system (2.5.5).

Some results on unique continuation properties

As we can see in the simple example, the unique continuation properties defined in Definition 2.5.2 hold for constant coefficient metrics. But we could also construct a counter-example such that this unique continuation property does not hold. In dimension 1, we assume that the metric $g = c(x)dx^2$. Then $\Delta_g = \frac{1}{c}\frac{d^2}{dx^2} - \frac{c'}{2c^2}\frac{d}{dx}$. Fix the open interval $(0,\pi)$ and the subinterval $(a,b) \subset (0,\pi)(a > \frac{\pi}{2})$. Now we consider the unique continuation problem:

$$\begin{cases} u_1'' = -\lambda^2 u_1, \\ \Delta_g u_2 = -\lambda^2 u_2, \\ u_1 + u_2 = 0 \text{ in } (a,b), \\ u_1, u_2 \in H_0^1((0,\pi)). \end{cases} \tag{2.5.13}$$

We have the following result:

Theorem 2.5.5. *There exists a smooth Riemannian metric $g = c(x)dx^2$, and two eigenfunctions u_1, u_2 of Δ_g and $\frac{d^2}{dx^2}$ on $(0,\pi)$ associated with eigenvalue 1 such that $u_1 + u_2 = 0$, in $(a,b) \subset (0,\pi)$ and $u_1 + u_2 \not\equiv 0$ in $(0,\pi)$.*

The readers can find the detailed construction of this counter-example in the section 3.5. Looking at the system 2.5.13, we consider the intersection of the spectrum of two Laplacians with different metrics. Let us define the space of all smooth metrics on the open interval $(0,\pi)$ by $\mathcal{M}^1$. We are able to prove the following proposition:

Proposition 2.5.6. *In dimension 1, suppose that we fix the Laplacian $\Delta = \frac{d^2}{dx^2}$ in $(0,\pi)$ with its spectrum $\sigma(\Delta)$. Then the set $\mathcal{G}_{uc} = \{g \in \mathcal{M}^1 : \sigma(\Delta_g) \cap \sigma(\Delta) = \emptyset\}$ is residual in $\mathcal{M}^1$.*

Roughly speaking, we are able to find "many" metrics in the sense of generic properties such that the spectrum of two Laplacians with different metrics are disjoint. Therefore, we obtain the following corollary immediately:

Corollary 2.5.7. *Fix $\Delta = \frac{d^2}{dx^2}$, for every metric $g \in \mathcal{G}_{uc}$, the system (2.5.13) has a unique solution $u_1 = u_2 = 0$.*

That is to say, the unique continuation property is true “genericly”. In addition, in dimension 2, we can also obtain the similar result:

Proposition 2.5.8. *In dimension 2, suppose that we fix one metric g_0 and the associated Laplacian Δ_{g_0} with its spectrum $\sigma(\Delta_{g_0})$. Then the set $\mathcal{G}_{uc} = \{g \in \mathcal{M}^2 : \sigma(\Delta_g) \cap \sigma(\Delta_{g_0}) = \emptyset\}$ is residual in $\mathcal{M}^2$.*

Here $\mathcal{M}^2$ is the space of all smooth metrics on the open domain $\Omega \subset \mathbb{R}^2$. And for proof details, we refer to the section 3.5.4.

Comments

There are two crucial parts in this proof. We need to get the microlocal information of each solution through the constraints on the sum of solutions. The other one is to prove the unique continuation property. In the first part, we mainly rely on the facts that the characteristic manifolds with different speeds are disjoint. Hence, in the high-frequency regime, we could distinguish every solution among the sum of them. For the second part, the main difficulty is that we have $n(n \geq 1)$ equations but with only one constraint to solve the unique continuation problem. In the constant coefficient case, the laplacians commute with each other. So we could apply the Δ for $n-1$ times to obtain $n-1$ constraints $\sum_i \Delta^k \phi_i = 0 (1 \leq k \leq n)$ in ω. Then we could reduce this problem into a unique continuation problem for a single equation. However, for general metrics, the laplacians do not commute with each other. Then this method does not work.

2.5.2 Coupled by a block-cascade structure

In this section, we mainly consider the Laplacian with constant coefficients. This is a joint work with Pierre Lissy. In this article, we proved the controllability of a coupled wave system with a single control and different speeds.

Motivations

To begin with, we introduce a simple example as follows:

$$\begin{cases} (\partial_t^2 - \Delta)u_1 + u_2 & = 0 \quad \text{in } (0,T)\times\Omega, \\ (\partial_t^2 - 2\Delta)u_2 + u_3 & = 0 \quad \text{in } (0,T)\times\Omega, \\ (\partial_t^2 - 2\Delta)u_3 & = f\mathbf{1}_\omega \quad \text{in } (0,T)\times\Omega, \end{cases} \tag{2.5.14}$$

with the Dirichlet boundary condition and some initial data, where f is a L^2 function supported in $(0,T)\times\omega$. Compared with (2.5.1), we consider a block-cascade coupling structure for the solutions. Notably, the control f is only acting directly on u_3, which itself acts on u_2 while u_1 is controlled through u_2.

For this example system, the controllability from zero is equivalent to the null controllability. Therefore, we begin with zero initial conditions. We first observe a regularity gap among the solutions, i.e. $(u_1, u_2, u_3) \in H^4 \times H^2 \times H^1$. In fact, since u_3 satisfies a wave equation with a source term $f \in L^1((0,T), L^2)$, it is classical that there exists a unique solution $u_3 \in C^1([0,T], H_0^1) \cap C^0([0,T], L^2)$. Since u_2 satisfies a wave equation with a source term $-u_3$, then $u_2 \in C^1([0,T], H^2) \cap C^0([0,T], H_0^1)$. For u_1, similarly, we obtain that $u_1 \in C^1([0,T], H^3) \cap C^0([0,T], H^2)$. Now, we need to state an extra regularity property for u_1. Applying the d'Alembert operator $\Box_2 = \partial_t^2 - 2\Delta$ on both sides of the equation of $\Box_1 u_1 = (\partial_t^2 - \Delta) u_1 = -u_2$, we obtain that

$$\Box_2 \Box_1 u_1 = -\Box_2 u_2.$$

Since $\Box_2 u_2 = -u_3$, then we obtain that $\Box_2 \Box_1 u_1 = u_3$. We consider that $\Box_2 u_1$ satisfies a wave equation with a source term u_3. Therefore, we know that $\Box_2 u_1 \in C^1([0,T], H^2) \cap C^0([0,T], H_0^1)$. Since $\Box_1 u_1 = -u_2 \in C^1([0,T], H^2) \cap C^0([0,T], H_0^1)$, we know that $\Delta u_1 = \Box_1 u_1 - \Box_2 u_1 \in C^1([0,T], H^2) \cap C^0([0,T], H_0^1)$. As a consequence, we know $u_1 \in C^1([0,T], H^4) \cap C^0([0,T], H^3)$. Hence, we notice a regularity gap $(u_1, u_2, u_3) \in H^4 \times H^2 \times H^1$. One can refer to [20] for a different proof.

In addition, with zero initial conditions, we also notice that there is a compatibility condition for this control problem, i.e. $(-\Delta)^2 u_1 + \Delta u_2 \in H_0^1$. In fact, let us first do some reformulation for the system. Define the transform $\mathcal{S}$ by

$$\mathcal{S} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

where

$$\begin{cases} v_1 = D_t^3 u_1, \\ v_2 = D_t u_2, \\ v_3 = u_3. \end{cases} \tag{2.5.15}$$

Moreover, (v_1, v_2, v_3) satisfies the following system:

$$\begin{cases} \Box_1 v_1 + D_t^2 v_2 = 0 \text{ in } (0,T) \times \Omega, \\ \Box_2 v_2 + D_t v_3 = 0 \text{ in } (0,T) \times \Omega, \\ \Box_2 v_3 = f \text{ in } (0,T) \times \Omega. \end{cases} \tag{2.5.16}$$

Using the identity

$$-D_t^2 = 2\Box_1 - \Box_2, \tag{2.5.17}$$

we obtain that

$$D_t^2 v_2 = -(2\Box_1 - \Box_2) v_2. \tag{2.5.18}$$

Using(2.5.18) in the first equation of (2.5.16), we also deduce that

$$\Box_1 (v_1 - 2v_2) - D_t v_3 = 0. \tag{2.5.19}$$

Now, let us define

$$y = D_t v_1 - 2D_t v_2. \tag{2.5.20}$$

Then, by (2.5.20) and (2.5.19), we obtain that

$$\Box_1 y - D_t^2 v_3 = 0. \tag{2.5.21}$$

We also remark that by using (2.5.17),

$$-D_t^2 v_3 = (2\Box_1 - \Box_2)v_3. \tag{2.5.22}$$

Hence, we deduce that

$$\Box_1(y + 2v_3) = f. \tag{2.5.23}$$

Let us now express y with respect to the original variables u_1, u_2, u_3. From (2.5.20), (2.5.15) and the first equation of (2.5.14), we obtain that

$$\begin{aligned} y &= D_t v_1 - 2D_t v_2 \\ &= D_t^4 u_1 - 2D_t^2 u_2 \\ &= D_t^2(D_t^2 u_1 - 2u_2) \\ &= D_t^2(-\Delta u_1 + u_2 - 2u_2) \\ &= D_t^2(-\Delta u_1 - u_2). \end{aligned}$$

Combining with the second equation of (2.5.14), we obtain

$$y = (-\Delta)^2 u_1 + \Delta u_2 - u_3.$$

Now, we define

$$\tilde{y} = y + 2u_3.$$

Then, $\tilde{y}$ satisfies

$$\Box_1 \tilde{y} = f. \tag{2.5.24}$$

With zero initial conditions, we obtain that $\tilde{y} \in H_0^1$, i,e, $(-\Delta)^2 u_1 + \Delta u_2 \in H_0^1$. Considering the regularity of u_1 and u_2, we know that $(u_1, u_2) \in H^4 \times H^2$. Hence, we can only obtain $(-\Delta)^2 u_1 + \Delta u_2 \in L^2$. Therefore, we notice a regularity gap between these two conditions. This gap implies that when we choose the appropriate state spaces, we need to consider not only the regularity of the solutions but also the compatibility conditions associated with the coupling structure. This is quite different from the system without coupling, and even different from the wave system coupled by the same speed or coupled parabolic systems. To our knowledge, this is one feature for such kind of coupled wave systems. This motivates us to consider a more general system with the same type of coupling structure.

The controllability for a wave system coupled with different speeds

We aim to deal with some controllability properties of the following type of coupled wave systems:

$$\begin{cases} (\partial_t^2 - D\Delta)U + AU & = \hat{b} f \mathbf{1}_\omega & \text{in } (0,T)\times\Omega, \\ U & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (U, \partial_t U)|_{t=0} & = (U^0, U^1) & \text{in } \Omega, \end{cases} \tag{2.5.25}$$

with here

$$D = \begin{pmatrix} d_1 Id_{n_1} & 0 \\ 0 & d_2 Id_{n_2} \end{pmatrix}_{n\times n}, A = \begin{pmatrix} 0 & A_1 \\ 0 & A_2 \end{pmatrix}_{n\times n}, \text{ and } \hat{b} = \begin{pmatrix} 0 \\ b \end{pmatrix}_{n\times 1}, \tag{2.5.26}$$

where $n = n_1 + n_2$ and $d_1 \neq d_2$. $A_1 \in \mathcal{M}_{n_1,n_2}(\mathbb{R})$ and $A_2 \in \mathcal{M}_{n_2}(\mathbb{R})$ are two given coupling matrices and $b \in \mathbb{R}^{n_2}$.

For $j = 1, 2$, we use $U_j = \begin{pmatrix} u_1^j \\ \vdots \\ u_{n_j}^j \end{pmatrix}$ to denote the solutions corresponding to the speed d_j respectively. Let us emphasize the following important and crucial properties of System (2.5.25): all coefficients are constant, the coupling is in a block-cascade structure (notably, the control f is only acting directly on U_2, which itself acts on U_1 through the matrix A_1), and we restrict to the case of a scalar control (*i.e.* $f \in L^2((0,T),\mathbb{R}^m)$ with $m = 1$).

Equivalent operator Kalman rank condition

In the following proposition, we give an equivalent statement of the operator Kalman rank condition associated with System (2.5.25), which is very specific to our particular coupling structure and the fact that we have a single control.

Proposition 2.5.9. *We use the same notations as in Definition 2.5.26. We denote by $\mathcal{K} = [-D\Delta + A|\hat{B}]$ the Kalman operator associated with System* (2.5.25). *Then, $Ker(\mathcal{K}^*) = \{0\}$ is equivalent to satisfying all the following conditions:*

1. $n_1 = 1$;
2. (A_2, B) *satisfies the usual Kalman rank condition (See Definition 2.2.3);*
3. *Assume that $A_1 = \alpha = (\alpha_1, \cdots, \alpha_{n_2})$. Then $\forall \lambda \in \sigma(-\Delta)$, α satisfies*

$$\alpha \left(\sum_{k=0}^{n_2-2} (d_1 - d_2)^k \lambda^k \sum_{j=k+1}^{n_2} a_j A_2^{j-1-k} + (d_1 - d_2)^{n_2-1} \lambda^{n_2-1} Id_{n_2} \right) \widehat{b} \neq 0, \tag{2.5.27}$$

where $(a_j)_{0\le j\le n_2}$ are the coefficients of the the characteristic polynomial of the matrix A_2, i.e. $\chi(X) = X^{n_2} + \sum_{j=0}^{n_2-1} a_j X^j$, with the convention that $a_{n_2} = 1$.

With this equivalent condition, we are able to simplify the system into

$$\begin{cases} \Box_1 u_1^1 + \sum_{j=1}^{s} \alpha_s u_j^2 = 0 \text{ in } (0,T)\times\Omega, \\ \Box_2 u_1^2 + u_2^2 = 0 \text{ in } (0,T)\times\Omega, \\ \vdots \\ \Box_2 u_{n_2-1}^2 + u_{n_2}^2 = 0 \text{ in } (0,T)\times\Omega, \\ \Box_2 u_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j} u_j^2 = f\mathbb{1}_\omega \text{ in } (0,T)\times\Omega, \\ u_1^1 = 0, u_j^2 = 0 \quad \text{on } (0,T)\times\partial\Omega, 1\le j\le n_2, \\ (u_1^1, u_1^2, \cdots, u_{n_2}^2)|_{t=0} = (u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) \text{ in } \Omega, \\ (\partial_t u_1^1, \partial_t u_1^2, \cdots, \partial_t u_{n_2}^2)|_{t=0} = (u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) \text{ in } \Omega. \end{cases} \tag{2.5.28}$$

Here we take $n_1 = 1$, $A_1 = (\alpha_1, \cdots, \alpha_s, 0, \cdots, 0)$ and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Appropriate state spaces

Since we consider the control problem in a domain Ω with boundary, it is natural for us to introduce the following Hilbert spaces $H_\Omega^s(\Delta)$.

Definition 2.5.10. *We denote by $(\beta_j^2)_{j\in\mathbb{N}^*}$ the non-decreasing sequence of (positive) eigenvalues of the Laplace operator $-\Delta$ with Dirichlet boundary condition, repeated with multiplicity, and $(e_j)_{j\in\mathbb{N}^*}$ an orthonormal basis of $L^2(\Omega)$ made of eigenfunctions associated with $(\beta_j^2)_{j\in\mathbb{N}^*}$:*

$$-\Delta e_j = \beta_j^2 e_j, \quad ||e_j||_{L^2} = 1.$$

For any $s\in\mathbb{R}$, we denote by $H^s(\Omega)$ the usual Sobolev space and by $H_\Omega^s(\Delta)$ the Hilbert space defined by

$$H_\Omega^s(\Delta) = \{u = \sum_{j\in\mathbb{N}^*} a_j e_j; \sum_{j\in\mathbb{N}^*} (1+\beta_j^2)^s |a_j|^2 < \infty\}. \tag{2.5.29}$$

Under this particular structure of coupling, we introduce appropriate compatibility conditions for System (4.1.6). For $r = 0, 1$, and $(u, v_1, \cdots, v_{n_2}) \in$

$H_\Omega^{n_2-s+2+r}(\Delta_D) \times H_\Omega^{n_2-1+r}(\Delta_D) \times \cdots \times H_\Omega^r(\Delta_D)$, let us define a special function U^r_{comp} by

$$\begin{aligned} U^r_{comp} = \Big(& (-d_1\Delta)^{n_2-s+1}u \\ & + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l} (-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l} v_{j+l} \\ & + \sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k} \frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}} \binom{n_2-s-k}{l} (-d_2\Delta)^{n_2-s-k-l} v_{j+k+l} \Big). \end{aligned} \tag{2.5.30}$$

Using this special function U^r_{comp}, let us denote by $\mathcal{H}^s_r$ the following space:

$$\begin{aligned} \mathcal{H}^s_r = \{ & (u, v_1, \cdots, v_{n_2}) \in H_\Omega^{n_2-s+2+r}(\Delta_D) \times H_\Omega^{n_2-1+r}(\Delta_D) \times \cdots \times H_\Omega^r(\Delta_D) \\ & \text{s.t. } U^r_{comp} \in H^r_\Omega(\Delta_D) \}. \end{aligned} \tag{2.5.31}$$

Definition 2.5.11 (State space)**.** *The state space for System* (4.4.1) *is defined by*

$$\mathcal{H}_1 \times \mathcal{H}_0.$$

The two conditions

$$\begin{aligned} & U^1_{comp}(u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) \in H^1_\Omega(\Delta_D), \\ & U^0_{comp}(u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) \in H^0_\Omega(\Delta_D) \end{aligned}$$

are called the compatibility conditions for the controllability of System (4.4.1)*.*

With these well-prepared spaces, we obtain the following result:

Theorem 2.5.12. *Given $T > 0$, suppose that:*

1. *(ω, T, p_{d_i}) satisfies GCC, $i = 1, 2$.*
2. *Ω has no infinite order of tangential contact with the boundary.*
3. *The Kalman operator $\mathcal{K} = [-D\Delta + A|\hat{B}]$ associated with System* (4.1.1) *satisfies the operator Kalman rank condition, i.e. $Ker(\mathcal{K}^*) = \{0\}$.*

Then the system (4.1.1) is exactly controllable.

We prove the above theorem within three steps.

1. At the first, we simplify the system (2.5.25), using the Brunovský normal form. This is based on the Proposition 2.5.9 and we only need to prove the exact/null controllability for the simplified system (2.5.28).

2. At the second step, we use the iteration schemes to obtain the compatibility conditions associated with the coupling structure in the system (2.5.25). Therefore, we prepare the appropriate state spaces for the controllability of the system (2.5.28).

3. In the final step, we use Hilbert uniqueness method to derive the observability inequality and then we follow the similar procedure as we did in the previous section 2.5.1. We establish a weak observability inequality and prove this weak observability inequality by the argument of contradiction and the propagation of the defect measures for systems. At last, the unique continuation property is given by the Kalman rank condition.

Comments

The main difficulty here is that the block-cascade coupling increases the difficulty for us to describe the proper Hilbert spaces for the states. As we presented in the example, only describing the regularity of each solution is not enough to construct the state spaces. The crucial part in the proof of the main result is to obtain the compatibility conditions associated with the coupling structure. The coupling with different speeds play a very important role in this problem.

2.5.3 Some comments on further developments

Based on the previous results, we already solved two special cases of the interior controllability for the coupled wave systems. Then, we could think about some more general coupling structures. For example, in the system (2.5.25) with

$$D = \begin{pmatrix} d_1 Id_{n_1} & 0 \\ 0 & d_2 Id_{n_2} \end{pmatrix}_{n\times n}, A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{n\times n}, \text{ and } \hat{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_{n\times m}, \tag{2.5.32}$$

In this case, the coupling is in a very general form and moreover, we consider some multi-control functions (*i.e.* $f \in L^2((0,T),\mathbb{R}^m)$ with $m > 1$). In such example, there are two types of difficulties. The first one is to find a algebraic equivalent condition for the abstract operator Kalman rank condition to simplify the coupling structure. The second one is to construct the appropriate state spaces, especially find the compatibility conditions under this setting.

Chapter 3

Simultaneous Control of Wave Systems

3.1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, be a bounded, and smooth domain. For positive constants α and β, let $k_{ij}(x) : \Omega \to \mathbb{R}$, $1 \le i, j \le d$ be smooth functions which satisfy:

$$k_{ij}(x) = k_{ji}(x), \alpha|\xi|^2 \le \sum_{1 \le i,j \le d} k_{ij}(x)\xi_i\xi_j \le \beta|\xi|^2, \forall x \in \Omega, \forall \xi \in \mathbb{R}^d. \tag{3.1.1}$$

Define $K(x)$ to be the symmetric positive definite matrix of coefficients $k_{ij}(x)$. Moreover, we define the density function $\kappa(x) = \frac{1}{\sqrt{\det(K(x))}}$. We also define the Laplacian by $\Delta_K = \frac{1}{\kappa(x)} div(\kappa(x)K\nabla\cdot)$ on Ω and the d'Alembert operator $\Box_K = \partial_t^2 - \Delta_K$ on $\mathbb{R}_t \times \Omega$. We assume that ω is a nonempty open subset of Ω. We consider the interior simultaneous controllability problem for the following wave system:

$$\begin{cases} \Box_{K_1} u_1 = b_1 f \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ \Box_{K_2} u_2 = b_2 f \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ \vdots \\ \Box_{K_n} u_n = b_n f \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ u_j = 0 \quad \text{on } (0,T) \times \partial\Omega, 1 \le j \le n, \\ u_j(0,x) = u_j^0(x), \quad \partial_t u_j(0,x) = u_j^1(x), 1 \le j \le n. \end{cases} \tag{3.1.2}$$

Here, we choose $K_i (1 \le i \le n)$ to be n different symmetric positive definite matrices. The state of the system is $(u_1, \partial_t u_1, \cdots, u_n, \partial_t u_n)$ and f is our control function. b_i are n nonzero constant coefficients. In this chapter, we mainly consider the exact controllability for the system Equation (3.1.2) given by the following definition.

Definition 3.1.1 (Exact Controllability)**.** *We say that the system Equation* (3.1.2) *is exactly controllable if for any initial data* $(u_1^0, u_1^1, \cdots, u_n^0, u_n^1) \in (H_0^1(\Omega) \times L^2(\Omega))^n$ *and any target data* $(U_1^0, U_1^1, \cdots, U_n^0, U_n^1) \in (H_0^1(\Omega) \times L^2(\Omega))^n$*, there exists a control function* $f \in L^2((0,T) \times \omega)$ *such that the solution of the system Equation* (3.1.2) *with initial data* $(u_1, \partial_t u_1, \cdots, u_n, \partial_t u_n)|_{t=0} = (u_1^0, \cdots, u_n^1)$ *satisfies* $(u_1, \partial_t u_1, \cdots, u_n, \partial_t u_n)|_{t=T}$ $= (U_1^0, \cdots, U_n^1)$.

Moreover, we also consider the partial exact controllability for the system Equation (3.1.2) given by the following definition.

Definition 3.1.2. *Let* Π *be a projection operator of* $(H_0^1(\Omega) \times L^2(\Omega))^n$*. We say that the system Equation* (3.1.2) *is* $\Pi-$*exactly controllable if for any initial data* $(u_1^0, u_1^1, \cdots, u_n^0, u_n^1) \in (H_0^1(\Omega) \times L^2(\Omega))^n$ *and any target data* $(U_1^0, U_1^1, \cdots, U_n^0, U_n^1) \in (H_0^1(\Omega) \times L^2(\Omega))^n$*, there exists a control function* $f \in L^2((0,T) \times \omega)$ *such that the solution of Equation* (3.1.2) *with initial data* $(u_1, \partial_t u_1, \cdots, u_n, \partial_t u_n)|_{t=0} = (u_1^0, u_1^1, \cdots, u_n^0, u_n^1)$ *satisfies*

$$\Pi(u_1, \partial_t u_1, \cdots, u_n, \partial_t u_n)|_{t=T} = \Pi(U_1^0, U_1^1, \cdots, U_n^0, U_n^1).$$

If we only impose that $\Pi(u_1, \partial_t u_1, \cdots, u_n, \partial_t u_n)|_{t=T} = 0$*, we say that the system Equation* (3.1.2) *is* $\Pi-$*null controllable.*

Proposition 3.1.3. *For System Equation* (3.1.2)*, the* $\Pi-$*null controllability is equivalent to the* $\Pi-$*exact controllability.*

Proof. We follow closely the proof of [17, Theorem 2.41]. It is clear that ($\Pi-$exact controllability) $\Longrightarrow$ ($\Pi-$null controllability). So we focus on the proof of the converse. We define the operator

$$\mathscr{A} = \begin{pmatrix} 0 & -1 & \cdots & 0 \\ -\Delta_{K_1} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & -1 \\ 0 & 0 & -\Delta_{K_n} & 0 \end{pmatrix}. \tag{3.1.3}$$

The system Equation (3.1.2) is equivalent to

$$\partial_t y = -\mathscr{A} y + \tilde{B} f \mathbf{1}_{(0,T)}(t) \mathbf{1}_\omega(x), y|_{t=0} = y(0), \tag{3.1.4}$$

where

$$y = \begin{pmatrix} u_1 \\ \partial_t u_1 \\ \vdots \\ u_n \\ \partial_t u_n \end{pmatrix}, \quad y(0) = \begin{pmatrix} u_1^0 \\ u_1^1 \\ \vdots \\ u_n^0 \\ u_n^1 \end{pmatrix} \quad \text{and } \tilde{B} = \begin{pmatrix} 0 \\ b_1 \\ \vdots \\ 0 \\ b_n \end{pmatrix}.$$

Let us consider $S(t)$ the semi-group generated by $\mathscr{A}$. Let $y^0 \in (H_0^1(\Omega) \times L^2(\Omega))^n$ and $y^1 \in (H_0^1(\Omega) \times L^2(\Omega))^n$. Since the system Equation (3.1.2) is $\Pi-$null controllable, we obtain that there exists f such that the solution $\tilde{y}$ of the Cauchy problem

$$\partial_t \tilde{y} = -\mathscr{A}\tilde{y} + \tilde{B} f \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x), y|_{t=0} = y^0 - S(-T)y^1 \tag{3.1.5}$$

satisfies $\Pi\tilde{y}(T) = 0$. For the Cauchy problem

$$\partial_t y = -\mathscr{A} y + \tilde{B} f \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x), y|_{t=0} = y^0, \tag{3.1.6}$$

the solution y is given by

$$y(t) = \tilde{y}(t) + S(t-T)y^1, \forall t \in [0,T]. \tag{3.1.7}$$

Hence, we obtain that $y(T) = \tilde{y}(T) + y^1$. In particular, we know that $\Pi y(T) = \Pi y^1$ since $\Pi\tilde{y}(T) = 0$. We now obtain the $\Pi-$exact controllability for the system Equation (3.1.2). $\square$

According to the Hilbert Uniqueness Method of J.-L. Lions [38], the controllability property is equivalent to an observability inequality for the adjoint system. In particular, when we focus on our system Equation (3.1.2), the exact controllability is equivalent to proving the following observability inequality: $\exists C > 0$ such that for any solution of the adjoint system:

$$\begin{cases} \square_{K_1} v_1 = 0 \text{ in } (0,T) \times \Omega, \\ \square_{K_2} v_2 = 0 \text{ in } (0,T) \times \Omega, \\ \vdots \\ \square_{K_n} v_n = 0 \text{ in } (0,T) \times \Omega, \\ v_j = 0 \quad \text{on } (0,T) \times \partial\Omega, 1 \le j \le n, \\ v_j(0,x) = v_j^0(x), \quad \partial_t v_j(0,x) = v_j^1(x), 1 \le j \le n, \end{cases} \tag{3.1.8}$$

we have

$$C \int_0^T \int_\omega |b_1\kappa_1 v_1 + \cdots + b_n\kappa_n v_n|^2 dxdt \ge \sum_{i=1}^n (||v_i^0||_{L^2}^2 + ||v_i^1||_{H^{-1}}^2). \tag{3.1.9}$$

For the partial controllability, we have a similar result. The $\Pi-$exact controllability of the system Equation (3.1.2) is equivalent to proving the following observability inequality: $\exists C > 0$ such that for any solution of the adjoint system:

$$\begin{cases} \square_{K_1} v_1 = 0 \text{ in } (0,T) \times \Omega, \\ \square_{K_2} v_2 = 0 \text{ in } (0,T) \times \Omega, \\ \vdots \\ \square_{K_n} v_n = 0 \text{ in } (0,T) \times \Omega, \\ v_j = 0 \quad \text{on } (0,T) \times \partial\Omega, 1 \le j \le n, \\ (v_1(0,x), \partial_t v_1(0,x), \cdots, v_n(0,x)\partial_t v_n(0,x)) = \Pi^* V^0, \end{cases} \tag{3.1.10}$$

where $V^0 \in (L^2 \times H^{-1})^n$ and Π^* is the adjoint operator of the projector Π, we have

$$C\int_0^T\int_\omega |b_1\kappa_1 v_1 + \cdots + b_n\kappa_n v_n|^2 dxdt \geq ||\Pi^* V^0||^2_{(L^2\times H^{-1})^n}. \tag{3.1.11}$$

This is an easy consequence of Proposition 3.1.3, the conservation of energy for system Equation (3.1.2) and [7, Chapter 4, Proposition 2.1].

In order to study the observability inequality, a classical method is to follow the abstract three-step process initialized by Rauch and Taylor [46](see also [10]). It can be detailed as follows:

- Firstly, get the microlocal information on the observable region. Argue by contradiction to obtain different kinds of convergence in subdomain $(0,T)\times\omega$ and the whole domain $(0,T)\times\Omega$.
- Secondly, use microlocal defect measure (which is due to Gérard [23] and Tartar [47]), or propagation of singulaties theorem (see [26] Section 18.1) to prove a weak observability estimate:

$$\begin{aligned}&\sum_{i=1}^n(||v_i^0||^2_{L^2} + ||v_i^1||^2_{H^{-1}})\\ &\leq C(\int_0^T\int_\omega|\sum_{j=1}^n b_j\kappa_j v_j|^2 dxdt + \sum_{i=1}^n(||v_i^0||^2_{H^{-1}} + ||v_i^1||^2_{H^{-2}})).\end{aligned}$$

- Thirdly, use unique continuation properties of eigenfunctions to obtain the original observability inequality Equation (3.1.9).

For the high frequency estimates, a very natural condition is to assume that the control set satisfies the Geometric Control Condition(GCC).

Definition 3.1.4. *For $\omega \subset \Omega$ and $T > 0$, we shall say that the pair (ω, T, p_K) satisfies GCC if every general bicharacteristic of p_K meets ω in a time $t < T$, where p_K is the principal symbol of $\Box_K$.*

We will give the definition of bicharacteristics in Section 3.3. This condition was raised by Bardos, Lebeau, and Rauch [9] when they considered the controllability of a scalar wave equation and has now become a basic assumption for the controllability of wave equations. In [14], the authors show that the geometric control condition is a necessary and sufficient condition for the exact controllability of the wave equation with Dirichlet boundary conditions and continuous boundary control functions. In order to study the low frequencies, we need to introduce the notion of unique continuation of eigenfunctions.

Definition 3.1.5. *We say the system Equation* (3.1.2) *satisfies the unique continuation of eigenfunctions if the following property holds:* $\forall \lambda \in \mathbb{C}$*, the only solution* $(\phi_1, \cdots, \phi_n) \in (H_0^1(\Omega))^n$ *of*

$$\begin{cases} -\Delta_{K_1}\phi_1 = \lambda^2 \phi_1 \ \textit{in } \Omega, \\ -\Delta_{K_2}\phi_2 = \lambda^2 \phi_2 \ \textit{in } \Omega, \\ \cdots \\ -\Delta_{K_n}\phi_n = \lambda^2 \phi_n \ \textit{in } \Omega, \\ b_1\kappa_1\phi_1 + \cdots + b_n\kappa_n\phi_n = 0 \ \textit{in } \omega, \end{cases}$$

is the zero solution $(\phi_1, \cdots, \phi_n) \equiv 0$.

There is a large literature on the controllability and observability of the wave equations. Several techniques have been applied to derive observability inequalities in various situations. This chapter is mainly devoted to multi-speed wave systems coupled by the control functions only. For other interesting situations, we list some of the existing results and references:

- For single wave equation, it is by now well-known that Bardos, Lebeau, and Rauch [10] use microlocal analysis to prove the Equation (3.1.9)-type observability inequality for a scalar wave equation. Other approaches for proving it can also be found in the literature, for example, using multipliers [38, 29], using Carleman estimates [25, 11], or completely constructive proof [30], etc.

- Although we now have a better picture on the controllabilty of a single wave equation, the controllability of systems of wave equations is still not totally understood. To our knowledge, most of the references concern the case of systems with the same principal symbol. Alabau-Boussouira and Léautaud [5] studied the indirect controllability of two coupled wave equations, in which their controllability result was established using a multi-level energy method introduced in [2], and also used in [3, 4]. Liard and Lissy [37], Lissy and Zuazua [40] studied the observability and controllability of the coupled wave systems under the Kalman type rank condition. Moreover, we can find other controllability results for coupled wave systems, for example, Cui, Laurent, and Wang [19] studied the observability of wave equations coupled by first or zero order terms on a compact manifold. The microlocal defect measure when dealing with the single wave equation can also be extended to a system case. One can refer to Burq and Lebeau for the microlocal defect measure for systems [15].

- As for multi-speed case, Dehman, Le Roussau, and Léautaud considered two coupled wave equations with multi-speeds in [21]. More related work is given by Tebou [48], in which the author considered the simultaneous controllability of constant multi-speed wave system and derived some result in a semilinear setting in [49].

3.1.1 Plan of the chapter

The chapter is organized as follows. Our main results are in Section 3.2 and Section 3.3 is devoted to introducing some geometric preliminaries. We include the descriptions of the boundary points, and give the precise definition of general bicharacteristics and the order of tangential contact with the boundary.

In Section 3.4, we focus on the high frequency estimates. Subsection 3.4.1 is devoted to introducing the microlocal defect measure and its basic properties, which is also the main tool for our proof. Subsection 3.4.2 deals with the partial controllability, and Subsection 3.4.3 is aimed to recover the exact controllability result in the whole energy space of initial conditions with the help of the unique continuation properties of eigenfunctions. In these two sections, we prove the Theorem 3.2.1, and Theorem 3.2.5 respectively.

In Section 3.5, we plan to deal with low frequency estimates, mainly discussing about the unique continuation properties of eigenfunctions. Subsection 3.5.1 provides a counterexample to show that only assuming the hypotheses in Theorem 3.2.1 cannot ensure the unique continuation properties of eigenfunctions. Then, we add some stronger assumptions to obtain the unique continuation property. The first attempt is to require an analyticity condition, which is the example in Proposition 3.5.3. The other attempt is to require constant coefficients in Subsection 3.5.2 and Subsection 3.5.3, which is stated in Theorem 3.2.8. Subsection 3.5.4 is about generic properties of metrics which ensure the unique continuation in dimension 1 and 2.

In Section 3.6, we deal with the constant coefficient case with multiple control functions. We also discuss the corresponding Kalman rank condition in this setting.

In Section 3.7, we include the proof of the equivalent condition of the Kalman rank condition in the case of multiple control functions.

3.1.2 Ideas of the proof

In our chapter, we prove the controllability result by applying the Hilbert uniqueness method to prove the observability inequality of the adjoint system. In order to study the observability inequality, we always use an argument by contradiction. First, we try to prove a weak observability inequality by adding some low frequency part. To obtain the original observability inequality, we need to analyse the invisible solutions in the subdomain $\omega \times (0, T)$ by proving the unique continuation properties of eigenfunctions. In section 4, we discuss some generic properties.

We follow the ideas given by Uhlenbeck [51], using the transversality theorem to obtain generic properties.

3.2 Main results

In this chapter, we mainly study the exact controllability for the system Equation (3.1.2) and discuss the optimality of the given conditions. On the other hand, when we consider the constant coefficient case, we associate the controllability with the Kalman rank condition. Instead of considering the exact controllability, we can only consider the high frequency estimates to obtain a partial result. One can also see similar finite codimensional controllability results, for instance, in [19] and [41].

Theorem 3.2.1. *Given $T > 0$, suppose that:*

1. (ω, T, p_{K_i}) *satisfies GCC,* $i = 1, 2, \cdots, n$,
2. $K_1 > K_2 > \cdots > K_n$ *in* ω,
3. Ω *has no infinite order of tangential contact on the boundary.*

Then, there exists a finite dimensional subspace $E \subset (H_0^1(\Omega) \times L^2(\Omega))^n$ *such that the system Equation* (3.1.2) *is* $\mathbb{P}-$*exactly controllable, where* $\mathbb{P}$ *is the orthogonal projector on* $E^\perp$.

We will explain the concept of the order of contact in the Section 3.3.

Remark 3.2.2. *We say that* $K_1 > K_2$ *in* ω *if and only if* $\forall x \in \omega$, $\forall \xi \in \mathbb{R}^d$ *and* $\xi \neq 0$, $(\xi, K_1(x)\xi) > (\xi, K_2(x)\xi)$, *where* $(\cdot, \cdot)$ *denotes the inner product of* $\mathbb{R}^d$.

Remark 3.2.3. *The Assumption* (2) *can be generalized as follows: let* σ *be a permutation of* $\{1, 2, \cdots, n\}$, $K_{\sigma(1)} > K_{\sigma(2)} > \cdots > K_{\sigma(n)}$ *in* ω.

Remark 3.2.4. *The same result holds for the laplacian operator*

$$\Delta_{K,\kappa} = \frac{1}{\kappa(x)} div(\kappa(x) K(x) \nabla \cdot),$$

where we only assume that $\kappa \in C^\infty(\Omega)$ *without the restriction* $\kappa(x) = \frac{1}{\sqrt{\det(K(x))}}$.

To obtain the exact controllability, we need more assumptions on the low frequency part.

Theorem 3.2.5. *Given $T > 0$, suppose that:*

1. (ω, T, p_{K_i}) *satisfies GCC,* $i = 1, 2, \cdots, n$,
2. $K_1 > K_2 > \cdots > K_n$ *in* ω,
3. Ω *has no infinite order of tangential contact on the boundary,*
4. *The system Equation* (3.1.2) *satisfies the unique continuation property of eigenfunctions.*

Then the system Equation (3.1.2) *is exactly controllable in* $(H_0^1(\Omega) \times L^2(\Omega))^n$.

Now, we consider the particular case of constant coefficients. Define the diagonal matrix $D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. We use Δ to denote the canonical Laplace operator. Now we consider the simultaneous control problem for the system:

$$\partial_t^2 U - D\Delta U = Bf\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \tag{3.2.1}$$

where $U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$. This system can be written as

$$\begin{cases} (\partial_t^2 - d_1\Delta)u_1 = b_1 f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ \vdots \\ (\partial_t^2 - d_n\Delta)u_n = b_n f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ u_j = 0 \quad \text{on } (0,T) \times \partial\Omega, 1 \le j \le n, \\ u_j(0,x) = u_j^0(x), \quad \partial_t u_j(0,x) = u_j^1(x), 1 \le j \le n. \end{cases}$$

First, we introduce the Kalman rank condition for the system Equation (3.2.1).

Definition 3.2.6 (Kalman rank condition)**.** *Define* $[D|B] = [D^{n-1}B|\cdots|DB|B]$. *We say* (D, B) *satisfies the Kalman rank condition if and only if* $[D|B]$ *has full rank.*

Remark 3.2.7. *In our setting,* (D, B) *satisfies the Kalman rank condition if and only if all* d_j *are distinct and* $b_j \neq 0$, $1 \le j \le n$*(See [6, Remark 1.1]).*

Theorem 3.2.8. *Given* $T > 0$, *suppose that:*

1. (ω, T, p_{d_i}) *satisfies GCC,* $i = 1, \cdots, n$.
2. Ω *has no infinite order of tangential contact on the boundary.*

Then the system Equation (3.2.1) *is exactly controllable in* $(H_0^1(\Omega) \times L^2(\Omega))^n$ *if and only if* (D, B) *satisfies the Kalman rank condition.*

Remark 3.2.9. *Let* T_0 *be the controllability time corresponding to the wave equation with unit speed of propagation. Then the controllability time in the Theorem* 3.2.8 *satisfies* $T > T_0 \max\{\frac{1}{\sqrt{d_j}}; j = 1, 2, \cdots, n\}$.

In advance, we consider the case with multiple control functions $f_1, f_2, \cdots, f_m (1 \leq m \leq n)$. To be more specific, we consider the system:

$$\begin{cases} \partial_t^2 U - D\Delta U = BF\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ U|_{\partial\Omega} = 0, \\ (U, \partial_t U)|_{t=0} = (U^0, U^1). \end{cases} \tag{3.2.2}$$

where $D = diag(d_1, d_2, \cdots, d_n)$, $F = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$, and $B = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}$. We can also define the Kalman rank condition $rank[D|B] = n$. Here we recall that $[D|B] = (D^{n-1}B|D^{n-2}B|\cdots|DB|B)$. We have the following theorem:

Theorem 3.2.10. *Given* $T > 0$*, suppose that:*

1. (ω, T, p_{d_i}) *satisfies GCC,* $i = 1, \cdots, n$.

2. Ω *has no infinite order of contact on the boundary.*

Then the system Equation (3.2.2) *is exactly controllable if and only if* (D, B) *satisfies the Kalman rank condition.*

Remark 3.2.11. *Since all coefficients and geometries are smooth, the use of the microlocal defect measures could have been replaced by propagation of singularities arguments.*

3.3 Geometric Preliminaries

This part has many repeated contents as we have already presented in Section 2.3 of Chapter 1.

Let $B = \{y \in \mathbb{R}^d : |y| < 1\}$ be the unit ball in $\mathbb{R}^d$. In a tubular neighbourhood of the boundary, we can identify $M = \Omega \times \mathbb{R}_t$ locally as $[0, 1[\times B$. More precisely, for $z \in \overline{M} = \overline{\Omega} \times \mathbb{R}_t$, we note that $z = (x, y)$, where $x \in [0, 1[$ and $y \in B$ and $z \in \partial M = \partial\Omega \times \mathbb{R}_t$ if and only if $z = (0, y)$. Now we consider $R = R(x, y, D_y)$

which is a second order scalar, self-adjoint, classical, tangential and smooth pseudo-differential operator, defined in a neighbourhood of $[0,1] \times B$ with a real principal symbol $r(x,y,\eta)$, such that

$$\frac{\partial r}{\partial \eta} \neq 0 \text{ for } (x,y) \in [0,1[\times B \text{ and } \eta \neq 0. \tag{3.3.1}$$

Let $Q_0(x,y,D_y)$, $Q_1(x,y,D_y)$ be smooth classical tangential pseudo-differential operators defined in a neighbourhood of $[0,1] \times B$, of order 0 and 1, and principal symbols $q_0(x,y,\eta)$, $q_1(x,y,\eta)$, respectively. Denote $P = (\partial_x^2 + R)Id + Q_0\partial_x + Q_1$. The principal symbol of P is

$$p = -\xi^2 + r(x,y,\eta). \tag{3.3.2}$$

We use the usual notations TM and T^*M to denote the tangent bundle and cotangent bundle corresponding to M, with the canonical projection π

$$\pi : TM(\text{ or } T^*M) \to M.$$

Denote $r_0(y,\eta) = r(0,y,\eta)$. Then we can decompose $T^*\partial M$ into the disjoint union $\mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$, where

$$\mathcal{E} = \{r_0 < 0\}, \quad \mathcal{G} = \{r_0 = 0\}, \quad \mathcal{H} = \{r_0 > 0\}. \tag{3.3.3}$$

The sets $\mathcal{E}$, $\mathcal{G}$, $\mathcal{H}$ are called elliptic, glancing, and hyperbolic set, respectively. Define $\mathrm{Char(P)} = \{(x,y,\xi,\eta) \in T^*\mathbb{R}^{d+1}|_{\overline{M}} : \xi^2 = r(x,y,\xi,\eta)\}$ to be the characteristic manifold of P. For more details, see [15] and [13].

3.3.1 Generalised bicharacteristic flow

We begin with the definition of the Hamiltonian vector field. For a symplectic manifold S with local coordinates (z,ζ), a Hamiltonian vector field associated with a real valued smooth function f is defined by the expression:

$$H_f = \frac{\partial f}{\partial \zeta}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial \zeta}.$$

Considering the principal symbol p, we can also consider the associated Hamiltonian vector field H_p. The integral curve of this Hamiltonian H_p, denoted by γ, is called a bicharacteristic of p. Our next goal is to study the behavior of the bicharacteristic near the boundary. To describe the different phenomena when a bicharacteristic approaches the boundary, we need a more accurate decomposition of the glancing set $\mathcal{G}$. Let $r_1 = \partial_x r|_{x=0}$. Then we can define the decomposition $\mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j$, with

$$\begin{aligned}
\mathcal{G}^2 &= \{(y,\eta) : r_0(y,\eta) = 0, r_1(y,\eta) \neq 0\},\\
\mathcal{G}^3 &= \{(y,\eta) : r_0(y,\eta) = 0, r_1(y,\eta) = 0, H_{r_0}(r_1) \neq 0\},\\
&\vdots\\
\mathcal{G}^{k+3} &= \{(y,\eta) : r_0(y,\eta) = 0, H_{r_0}^j(r_1) = 0, \forall j \leq k, H_{r_0}^{k+1}(r_1) \neq 0\},\\
&\vdots\\
\mathcal{G}^{\infty} &= \{(y,\eta) : r_0(y,\eta) = 0, H_{r_0}^j(r_1) = 0, \forall j\}.
\end{aligned}$$

Here $H_{r_0}^j$ is just the vector field H_{r_0} composed j times. Moreover, for $\mathcal{G}^2$, we can define $\mathcal{G}^{2,\pm} = \{(y,\eta) : r_0(y,\eta) = 0, \pm r_1(y,\eta) > 0\}$. Thus $\mathcal{G}^2 = \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}$. For $\rho \in \mathcal{G}^{2,+}$, we say that ρ is a gliding point and for $\rho \in \mathcal{G}^{2,-}$, we say that ρ is a diffractive point. For $\rho \in \mathcal{G}^j$, $j \geq 2$, we say that a bicharacterisric of p tangentially contact the boundary $\{x = 0\} \times B$ with order j at the point ρ.

Consider a bicharacteristic $\gamma(s)$ with $\pi(\gamma(0)) \in M$ and $\pi(\gamma(s_0)) \in \partial M$ be the first point which touches the boundary. Then if $\gamma(s_0) \in \mathcal{H}$, we can define $\xi^{\pm}(\gamma(s_0)) = \pm\sqrt{r_0(\gamma(s_0))}$, which are the two different roots of $\xi^2 = r_0$ at the point $\gamma(s_0)$. Notice that the bicharacteristic with the direction ξ^- will leave the domain M while the bicharacteristic with the other direction ξ^+ will enter into the interior of M. This leads to a definition of the broken bicharacteristics(See [26] Section 24.2 for more details):

Definition 3.3.1. *A broken bicharacteristic of p is a map:*

$$s \in I \backslash D \mapsto \gamma(s) \in T^*M \backslash \{0\}$$

where I is an interval on $\mathbb{R}$ and D is a discrete subset, such that

1. *If J is an interval contained in $I \backslash D$, then for $s \in J \mapsto \gamma(s)$ is a bicharacteristic of p in M.*
2. *If $s \in D$, then the limits $\gamma(s^+)$ and $\gamma(s^-)$ exist and belongs to $T_z^*M \backslash \{0\}$ for some $z \in \partial M$, and the projections in $T_z^*\partial M \backslash \{0\}$ are the same hyperbolic point.*

If $\gamma(s_0) \in \mathcal{G}$, we have different situations. If $\gamma(s_0) \in \mathcal{G}^{2,+}$, then $\gamma(s)$, locally near s_0, passes transversally and enters into T^*M immediately. If $\gamma(s_0) \in \mathcal{G}^{2,-}$ or $\gamma(s_0) \in \mathcal{G}^k$ for some $k \geq 3$, then $\gamma(s)$ will continue inside $T^*\partial M$ and follow the Hamiltonian flow of H_{-r_0}. To be more precise, we have the definition of the generalized bicharacteristics(See [26] Section 24.3 for more details):

Definition 3.3.2. *A generalized bicharacteristic of p is a map:*

$$s \in I \backslash D \mapsto \gamma(s) \in T^*M \cup \mathcal{G}$$

where I is an interval on $\mathbb{R}$ and D is a discrete subset I such that $p \circ \gamma = 0$ and the following properties hold:

1. *$\gamma(s)$ is differentiable and $\frac{\mathrm{d}\gamma}{\mathrm{d}s} = H_p(\gamma(s))$ if $\gamma(s) \in T^*M$ or $\gamma(s) \in \mathcal{G}^{2,+}$.*
2. *Every $t \in D$ is isolated i.e. there exists $\epsilon > 0$ such that $\gamma(s) \in T^*\overline{M} \backslash T^*\partial M$ if $0 < |s-t| < \epsilon$, and the limits $\gamma(s^{\pm})$ are different points in the same hyperbolic fiber of $T^*\partial M$.*
3. *$\gamma(s)$ is differentiable and $\frac{\mathrm{d}\gamma}{\mathrm{d}s} = H_{-r_0}(\gamma(s))$ if $\gamma(s) \in \mathcal{G} \backslash \mathcal{G}^{2,+}$.*

Remark 3.3.3. *We denote the Melrose cotangent compressed bundle by ${}^bT^*\overline{M}$ and the associated canonical map by $j : T^*\overline{M} \mapsto {}^bT^*\overline{M}$. j is defined by*

$$j(x, y, \xi, \eta) = (x, y, x\xi, \eta).$$

Under this map j, one could see $\gamma(s)$ as a continuous flow on the compressed cotangent bundle ${}^bT^*\overline{M}$. This is the so-called Melrose-Sjöstrand flow.

From now on we always assume that there is no infinite tangential contact between the bicharacteristic of p and the boundary. This is in the meaning of the following definition:

Definition 3.3.4. *We say that there is no infinite contact between the bicharacteristics of p and the boundary if there exists $N \in \mathbb{N}$ such that the gliding set $\mathcal{G}$ satisfies*

$$\mathcal{G} = \bigcup_{j=2}^{N} \mathcal{G}^j.$$

It is well-known that under this hypothesis there exists a unique generalized bicharacteristic passing through any point. This means that the Melrose-Sjöstrand flow is globally well-defined. One can refer to [42] and [43] for the proof.

3.4 High Frequency Estimates

3.4.1 Microlocal defect measure

In this section, we introduce the microlocal defect measures based on the article by Gérard and Leichtnam [24] for Helmoltz equation and Burq [13] for wave equations.

Let $(u^k)_{k\in\mathbb{N}} \in L^2_{loc}(\mathbb{R}_t; L^2(\Omega))$ be a bounded sequence, converging weakly to 0 and such that

$$\begin{cases} Pu^k = o(1)_{H^{-1}}, \\ u^k|_{\partial M} = 0. \end{cases} \tag{3.4.1}$$

Let $\underline{u}_k$ be the extension by 0 across the boundary of Ω. Then the sequence $\underline{u}_k$ is bounded in $L^2_{loc}(\mathbb{R}_t; L^2(\mathbb{R}^d))$. Let $\mathcal{A}$ be the space of classical polyhomogeneous pseudo-differential operators of order 0 with compact support in $\mathbb{R}_t \times \mathbb{R}^d$ (i.e, $A = \varphi A \varphi$ for some $\varphi \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}^d)$). Let us denote by $\mathcal{M}^+$ the set of non negative Radon measures on $S^*(\mathbb{R}_t \times \mathbb{R}^d)$. From [13, Section 1], we have the existence of the microlocal defect measure as follows:

Proposition 3.4.1 (Existence of the microlocal defect measure)**.** *There exists a subsequence of* $(\underline{u}^k)$ *(still noted by* $(\underline{u}^k)$*) and* $\mu \in \mathcal{M}^+$ *such that*

$$\forall A \in \mathcal{A}, \quad \lim_{k\to\infty} (A\underline{u}^k, \underline{u}^k)_{L^2} = \langle \mu, \sigma(A) \rangle, \tag{3.4.2}$$

where $\sigma(A)$ *is the principal symbol of the operator* A *(which is a smooth function homogeneous of order* 2 *in the variable* ξ*, i.e. a function on* $S^*((\mathbb{R}_t \times \mathbb{R}^d))$*.*

Remark 3.4.2. *In general, the existence of the microlocal defect measure does not rely on the system Equation* (3.4.1)*. For any bounded sequence* u^k *in* L^2*, which is weakly convergent to* 0*, one is able to construct the microlocal defect measure associated with the sequence (see [13] for more details).*

Remark 3.4.3. *In the article [31], Lebeau constructed the microlocal defect measure in another approach (see [31, Appendice] for more details). In the article [15], Burq and Lebeau proved the similar existence result [15, Proposition 2.5] in a setting of systems, which can be seen as an extension of Proposition* 3.4.1

From [13, Théorème 15], we have the following proposition.

Proposition 3.4.4. *For the microlocal defect measure* μ *defined above associated with the system Equation* (3.4.1)*, we have the following properties.*

- *The measure* μ *is supported on the intersection of the characteristic manifold with* $\mathbb{R}_t \times \overline{\Omega}$*,*

$$supp(\mu) \subset \{(t, x, \tau, \xi); x \in \overline{M}, \tau^2 = {}^t\xi K(x)\xi\}. \tag{3.4.3}$$

- *The measure* μ *does not charge the hyperbolic points in* ∂M*,*

$$\mu = 0 \text{ on } \pi_b^{-1}(\mathcal{H}),$$

where $\pi_b : T^*(\mathbb{R}^{d+1}) \to {}^bT^*\overline{M}$ *(the Melrose cotangent compressed bundle).*

- *The measure μ is invariant by the generalised bicharacteristic flow.*

Remark 3.4.5. *Notice first that in [13, Section 3], the author considered the case of solutions to the wave equation at the energy level (bounded in H^1_{loc}, and hence was considering second order operators. However, it is easy to pass from H^1 to L^2 solutions by applying the operator ∂_t and conversely from L^2 to H^1 by applying the operator ∂_t^{-1}, i.e. if v is an L^2 solution, considering the solution u associated to $\big((-\Delta_D)^{-1}(\partial_t v \mid_{t=0}), v \mid_{t=0}\big)$, which of course satisfies $\partial_t u = v$. This procedure amounts to replacing the test operators of order 0 A by the test operator of order 2, $B = -\partial_t \circ A \circ \partial_t$, but since τ^2 does not vanish on the characteristic manifold, it is an elliptic factor which changes nothing.*

Remark 3.4.6. *Notice also that due to discontinuity of the generalised bicharacteristics when they reflect on the boundary at hyperbolic points (the points corresponding to the left and right limits at $s \in D$), in Definition 3.3.1, the generalised bicharacteristic flow is not well defined (there are two points above any points corresponding to $s \in D$). However, since the measure μ does not charge these hyperbolic points, this flow is well defined μ almost surely and the invariance property makes sense. Notice also that in [13, Appendice], weaker property than invariance (namely that the support is a union of generalised bicahracteristics) is proved. The general result follows from this weaker result by applying the strategy in [31]. In any case, for the purpose of the present chapter, the invariance of the support would suffice.*

3.4.2 Proof of the Theorem 3.2.1

Let $V = (v_1^0, v_1^1, \cdots, v_n^0, v_n^1)$. We introduce the following spaces:

- We define $\mathcal{K}_1 = (H_0^1(\Omega) \times L^2(\Omega))^n$ endowed with the norm

$$||V||^2_{\mathcal{K}_1} = \sum_{j=1}^n \int_\Omega (K_j \nabla v_j^0 \cdot \overline{\nabla v_j^0} + |v_i^1|^2)\kappa_i \,\mathrm{d}x.$$

- We define $\mathcal{K}_0 = (L^2(\Omega) \times H^{-1}(\Omega))^n$ endowed with the norm

$$||V||^2_{\mathcal{K}_0} = \sum_{i=1}^n \int_\Omega |v_i^0|^2 \kappa_i dx + < v_i^1, T_{K_i} v_i^1 >_{H^{-1}, H_0^1},$$

where

$$T_{K_i} : H^{-1}(\Omega) \to H_0^1(\Omega)$$
$$f \mapsto w$$

is defined as the unique solution $w \in H_0^1(\Omega)$ to $-\frac{1}{\kappa_i} div(\kappa_i K_i \nabla T_{K_i} w) = f$.

- We define $\mathcal{K}_{-1} = (H^{-1}(\Omega) \times D(-\Delta)')^n$ endowed with the norm

$$||V||^2_{\mathcal{K}_{-1}} = \sum_{i=1}^{n} < v_i^0, T_{K_i} v_i^0 >_{H^{-1},H_0^1} + < v_i^1, \tilde{T}_{K_i} v_i^1 >_{D(-\Delta_{K_i})^*, D(-\Delta_{K_i})},$$

  where $D(-\Delta)$ is the domain of the Laplacian operator with zero Dirichlet boundary condition and $D(-\Delta)'$ is its dual space, and

$$\begin{aligned} \tilde{T}_{K_i} : D(-\Delta)' &\to D(-\Delta) \\ \tilde{f} &\mapsto \tilde{w} \end{aligned}$$

  is defined as the unique solution $\tilde{w} \in D(-\Delta)$ to $(-\Delta_{K_i})^2 \tilde{T}_{K_i} \tilde{w} = \tilde{f}$.

Remark 3.4.7. *For any $j \in \{1, 2, \cdots, n\}$, $D(-\Delta_{K_j}) = D(-\Delta)$.*

Recall the considered control system:

$$\begin{cases} \Box_{K_1} u_1 = b_1 f \mathbf{1}_{(0,T)}(t) \mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ \Box_{K_2} u_2 = b_2 f \mathbf{1}_{(0,T)}(t) \mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ \vdots \\ \Box_{K_n} u_n = b_n f \mathbf{1}_{(0,T)}(t) \mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ u_j = 0 \quad \text{ on } (0,T) \times \partial\Omega, 1 \le j \le n, \\ (u_1, \partial_t u_1, \cdots, u_n, \partial_t u_n)|_{t=0} = U(0). \end{cases} \tag{3.4.4}$$

Consider the homogeneous system:

$$\begin{cases} \Box_{K_1} v_1^h = 0 \text{ in } (0,T) \times \Omega, \\ \Box_{K_2} v_2^h = 0 \text{ in } (0,T) \times \Omega, \\ \vdots \\ \Box_{K_n} v_n^h = 0 \text{ in } (0,T) \times \Omega, \\ v_j^h = 0 \quad \text{ on } (0,T) \times \partial\Omega, 1 \le j \le n, \\ (v_1^h, \partial_t v_1^h, \cdots, v_n^h, \partial_t v_n^h)|_{t=0} = V^h(0) \in \mathcal{K}_1. \end{cases} \tag{3.4.5}$$

Now, let us define

$$E = \{V^h(0) \in \mathcal{K}_1 : (b_1 \kappa_1 v_1^h + \cdots + b_n \kappa_n v_n^h)(t,x) = 0, \text{ for any } t \in (0,T), x \in \omega\}, \tag{3.4.6}$$

where $(v_1^h, \cdots, v_n^h)$ is the solution to the homogeneous system Equation (3.4.5). Hence, E is a closed subspace in $\mathcal{K}_1$. Denote the orthogonal projector operator $\mathbb{P} : \mathcal{K}_1 \to E^\perp$. And the adjoint system of System Equation (3.4.4) is the following system:

$$\begin{cases} \Box_{K_1} v_1 = 0 \text{ in } (0,T)\times\Omega, \\ \Box_{K_2} v_2 = 0 \text{ in } (0,T)\times\Omega, \\ \vdots \\ \Box_{K_n} v_n = 0 \text{ in } (0,T)\times\Omega, \\ v_j = 0 \quad \text{on } (0,T)\times\partial\Omega, 1\le j\le n, \\ (v_1, \partial_t v_1, \cdots, v_n, \partial_t v_n)|_{t=0} = \mathbb{P}^* V(0) \in \mathcal{K}_0. \end{cases} \tag{3.4.7}$$

Using inequality Equation (3.1.11), the $\mathbb{P}-$exactly controllability of the system Equation (3.4.4) is equivalent to proving the following observability inequality:

$$C\int_0^T\int_\omega |b_1\kappa_1 v_1 + \cdots + b_n\kappa_n v_n|^2 dxdt \ge ||\mathbb{P}^* V(0)||^2_{\mathcal{K}_0}, \tag{3.4.8}$$

where $(v_1, \cdots, v_n)$ is the solution to the adjoint system Equation (3.4.7).

Step 1: Establish a weak observability inequality

First we want to prove a weak inequality:

$$||\mathbb{P}^* V(0)||^2_{\mathcal{K}_0} \le C\left(\int_0^T\int_\omega |b_1\kappa_1 v_1 + \cdots + b_n\kappa_n v_n|^2 dxdt + ||\mathbb{P}^* V(0)||^2_{\mathcal{K}_{-1}}\right), \tag{3.4.9}$$

If the above inequality was false, we could get a sequence $(\mathbb{P}^*\widetilde{V}_0^k)_{k\in\mathbb{N}}$ such that

$$||\mathbb{P}^*\widetilde{V}_0^k||^2_{\mathcal{K}_0} = 1, \tag{3.4.10}$$

$$\int_0^T\int_\omega |b_1\kappa_1 v_1^k + \cdots + b_n\kappa_n v_n^k|^2 dxdt \to 0, k\to\infty, \tag{3.4.11}$$

and

$$||\mathbb{P}^*\widetilde{V}_0^k||^2_{\mathcal{K}_{-1}} \to 0, k\to\infty. \tag{3.4.12}$$

Here we use $v_i^k (1\le i\le n)$ to denote the corresponding solution of the system Equation (3.4.7) with the initial data $\mathbb{P}^*\widetilde{V}_0^k$. Hence, we obtain n bounded sequences $\{v_i^k\}_{k\in\mathbb{N}} (1\le i\le n)$. Let μ_i be the defect measure associated to the sequence $\{v_i^k\}_{k\in\mathbb{N}}$, by the construction in Subsection 3.4.1. Notice that in these constructions, each sequence $\{v_i^k\}_{k\in\mathbb{N}}$ is solution to a particular wave equation

$$\Box_{K_i} v_i^k = 0, v_i^k\ |_{\partial\Omega} = 0$$

and in Section 3.3 this corresponds to different principal symbols p_i, different sets $\mathcal{G}_i, \mathcal{H}_i, \mathcal{E}_i$ and different generalised bicharacteristic γ_i.

From the definition of the measures, we obtain

$$\forall A \in \mathcal{A}, \quad \langle \mu_i, \sigma(A) \rangle = \lim_{k \to \infty} (A\underline{v}_i^k, \underline{v}_i^k)_{L^2},$$

where $\underline{v}_i^k$ is the extension by 0 across the boundary of Ω. From Proposition 3.4.4 we have

Lemma 3.4.8. *Each measure μ_i is supported on the characteristic manifold*

$$\mathrm{Char}(p_i) = \{(t, x, \tau, \xi) \in T^*\mathbb{R} \times \mathbb{R}^d \,|_{\overline{\Omega}}; \tau^2 = {}^t\xi K_i(x)\xi\}$$

and is invariant along the generalised bicharacteristic flow associated to the symbol $p_i ={}^t \xi K_i(x)\xi - \tau^2$

Lemma 3.4.9. *The measures μ_i and μ_l are mutually singular in $(0, T) \times \omega$, for $i \neq l$.*

Remark 3.4.10. *We recall that two measures μ and ν are singular if there exists a measurable set A such that $\mu(A) = 0$ and $\nu(A^c) = 0$.*

Proof. This follows easily from Lemma 3.4.8 and the assumption 2 in Theorem 3.2.5, which implies that over ω, the two characteristic manifolds $\mathrm{Char}(p_i)$ and $\mathrm{Char}(p_l)$ are disjoint. □

Lemma 3.4.11. *For $A \in \mathcal{A}$ with the compact support in $(0, T) \times \omega$, we obtain that for $i \neq l$:*

$$\limsup_{k \to \infty} |(A\underline{v}_i^k, \underline{v}_l^k)_{L^2}| = 0. \tag{3.4.13}$$

Proof. For $\forall (t, x) \in (0, T) \times \omega$, we have that

$$\mathrm{Char}(p_i) \cap \mathrm{Char}(p_l) = \{0\}, i \neq l.$$

Then we choose a cut-off function $\beta_i \in C^\infty(T^*\mathbb{R} \times \mathbb{R}^d)$ homogeneous of degree 0 for $|(\tau, \xi)| \geq 1$, with compact support in $(0, T) \times \omega$ such that

$$\beta_i|_{\mathrm{Char}(p_i)} = 1, \beta_i|_{\mathrm{Char}(p_l)} = 0, \text{ and } 0 \leq \beta_i \leq 1.$$

Since $A \in \mathcal{A}$ with the compact support in $(0, T) \times \omega$, for some $\varphi \in C_0^\infty((0, T) \times \omega)$, we have that $A = \varphi A \varphi$. We choose $\tilde{\varphi} \in C_0^\infty((0, T) \times \omega)$ such that $\tilde{\varphi}|_{\mathrm{supp}(\varphi)} = 1$ i.e, $\tilde{\varphi}\varphi = \varphi$. Now let us consider the $(A\underline{v}_i^k, \underline{v}_l^k)_{L^2}$. First, we have that

$$\begin{aligned}
(A\underline{v}_i^k, \underline{v}_l^k)_{L^2} &= (\varphi A \varphi \underline{v}_i^k, \underline{v}_l^k)_{L^2} \\
&= (\varphi A \varphi \underline{v}_i^k, \tilde{\varphi} \underline{v}_l^k)_{L^2} \\
&= ((1 - \mathrm{Op}(\beta_i))\varphi A \varphi \underline{v}_i^k, \tilde{\varphi} \underline{v}_l^k)_{L^2} + (\mathrm{Op}(\beta_i)\varphi A \varphi \underline{v}_i^k, \tilde{\varphi} \underline{v}_l^k)_{L^2}.
\end{aligned}$$

For the first term $((1-\mathrm{Op}(\beta_i))\varphi A\varphi \underline{v}_i^k, \tilde{\varphi}\underline{v}_l^k)_{L^2}$, by the Cauchy-Schwarz inequality, therefore we obtain that

$$|((1-\mathrm{Op}(\beta_i))\varphi A\varphi \underline{v}_i^k, \tilde{\varphi}\underline{v}_l^k)_{L^2}| \leq ||(1-\mathrm{Op}(\beta_i))\varphi A\varphi \underline{v}_i^k||_{L^2}||\tilde{\varphi}\underline{v}_l^k||_{L^2}$$

As we know that $\{\underline{v}_l^k\}$ is bounded in $L^2_{loc}(\mathbb{R}_t \times \mathbb{R}^d)$, there exists a constant C such that

$$||\tilde{\varphi}\underline{v}_l^k||^2_{L^2} = (\tilde{\varphi}\underline{v}_l^k, \tilde{\varphi}\underline{v}_l^k)_{L^2} \leq C.$$

From the definition of the measure μ_i, we obtain

$$\begin{aligned}\lim_{k\to\infty} ||(1-\mathrm{Op}(\beta_i))\varphi A\varphi \underline{v}_i^k||^2_{L^2} &= \lim_{k\to\infty}((1-\mathrm{Op}(\beta_i))\varphi A\varphi \underline{v}_i^k, (1-\mathrm{Op}(\beta_i))\varphi A\varphi \underline{v}_i^k)_{L^2}\\ &= \langle \mu_i, (1-\beta_i)^2\varphi^4|\sigma(A)|^2\rangle.\end{aligned}$$

From Lemma 4.2.9, we have that supp $(\mu_i) \subset \mathrm{Char}(p_i)$. In addition, by the choice of β_i, we know that $1-\beta_i \equiv 0$ on supp (μ_i), which implies that $\langle \mu_i, (1-\beta_i)^2\varphi^4|\sigma(A)|^2\rangle = 0$. Hence, we obtain

$$\limsup_{k\to\infty} |((1-\mathrm{Op}(\beta_i))\varphi A\varphi \underline{v}_i^k, \tilde{\varphi}\underline{v}_l^k)_{L^2}| = 0. \tag{3.4.14}$$

The other term $(\mathrm{Op}(\beta_i)\varphi A\varphi \underline{v}_i^k, \tilde{\varphi}\underline{v}_l^k)_{L^2} = (\underline{v}_i^k, \varphi A^*\varphi \mathrm{Op}(\beta_i)^*\tilde{\varphi}\underline{v}_l^k)_{L^2}$ is dealt with similarly by exchanging i and l. □

Now let us come back to the proof of the weak observability inequality Equation (3.4.9). By the assumption Equation (3.4.11), We know that

$$\int_0^T \int_\omega |b_1\kappa_1 v_1^k + \cdots + b_n\kappa_n v_n^k|^2 dxdt \to 0,$$

for $\chi \in C_0^\infty(\omega \times (0,T))$, and we would like to obtain:

$$\sum_{1\leq i,l\leq n} \langle \chi b_i\kappa_i v_i^k, \chi b_l\kappa_l v_l^k\rangle \to 0, \text{ as } k\to\infty.$$

According to Lemma 3.4.11, we know that for $i \neq l$,

$$\limsup_{k\to\infty} |\langle \chi b_i\kappa_i v_i^k, \chi b_l\kappa_l v_l^k\rangle| = 0. \tag{3.4.15}$$

As a consequence, we know that

$$\limsup_{k\to\infty} \Sigma_{i=1}^n \langle \chi b_i\kappa_i v_i^k, \chi b_i\kappa_i v_i^k\rangle = 0. \tag{3.4.16}$$

Using again the definition of the measure μ_i, we obtain the following:

$$0 \leq \langle \mu_i, (\chi b_i \kappa_i)^2 \rangle = \lim_{k\to\infty} \langle \chi b_i \kappa_i v_i^k, \chi b_i \kappa_i v_i^k \rangle \leq \limsup_{k\to\infty} \Sigma_{i=1}^n \langle \chi b_i \kappa_i v_i^k, \chi b_i \kappa_i v_i^k \rangle = 0. \tag{3.4.17}$$

Thus, we know that

$$\mu_i|_{\omega\times(0,T)} = 0.$$

Since μ_i is invariant along the general bicharacteristics of p_{K_i} (by Lemma 3.4.8), combining with GCC, we know that $\mu_i \equiv 0$. Since $\mu_i = 0$, we have $v_i^k \to 0$ strongly in $L^2_{loc}((0,T)\times\Omega)$. Now we have to estimate $||\partial_t v_1^k(0)||_{H^{-1}}$. Let $\chi \in C_0^\infty((0,T))$. Multiply the equation

$$\Box_{K_1} v_1 = 0$$

by $T_{K_1}(\chi^2 v_1^k)$ and then integrate on $(0,T)\times\Omega$. We obtain that

$$\begin{aligned}
0 &= \int_0^T \int_\Omega \Box_{K_1} v_1^k \cdot \overline{T_{K_1}(\chi^2 v_1^k)} \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega v_1^k \cdot \overline{(-\Delta_{K_1}) T_{K_1}(\chi^2 v_1^k)} \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \partial_t v_1 \cdot \overline{T_{K_1}(\partial_t(\chi^2) v_1^k)} \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^T ||\chi \partial_t v_1^k||^2_{H^{-1}} \\
&= ||\chi v_1^k||^2_{L^2} - \int_0^T ||\chi \partial_t v_1^k||^2_{H^{-1}} + \int_0^T \int_\Omega v_1^k \cdot \overline{T_{K_1}(\partial_t^2(\chi^2) v_1^k + \partial_t(\chi^2)\partial_t v_1^k)} \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.4.18}$$

For the term $\int_0^T \int_\Omega v_1^k \cdot \overline{T_{K_1}(\partial_t^2(\chi^2) v_1^k + \partial_t(\chi^2)\partial_t v_1^k)} \,\mathrm{d}x\,\mathrm{d}t$, we know that $v_1^k \to 0$ strongly in $L^2_{loc}((0,T)\times\Omega)$ and $T_{K_1}(\partial_t^2(\chi^2) v_1^k + \partial_t(\chi^2)\partial_t v_1^k)$ is bounded in L^2. Thus, up to a subsequence, it tends to 0 as $k \to \infty$. Hence, we obtain that:

$$\int_0^T ||\chi \partial_t v_1^k||^2_{H^{-1}} \to 0, \text{ as } k \to \infty.$$

So for all $0 < t_1 < t_2 < T$,

$$\int_{t_1}^{t_2} ||\partial_t v_1^k(t)||^2_{H^{-1}} dt \to 0.$$

So for almost every $t \in]t_1, t_2[$, $||\partial_t v_1^k(t)||^2_{H^{-1}} + ||v_1^k(t)||^2_{L^2} \to 0$. Then by the backward well-posedness, we can conclude:

$$||\partial_t v_1^k(0)||^2_{H^{-1}} + ||v_1^k(0)||^2_{L^2} \to 0.$$

The same reasoning holds for v_j^k, $2 \leq j \leq n$. This gives a contradiction with Equation (3.4.10), which proves the weak observability inequality Equation (3.4.9).

Remark 3.4.12. *Let us denote the energy $E(v_j^k)(t)$ by $E(v_j^k)(t) = ||\partial_t v_j^k(t)||_{H^{-1}}^2 + ||v_j^k(t)||_{L^2}^2$. In fact, each v_j^k satisfies a conservative system. Hence, we obtain*

$$E(v_j^k)(0) = E(v_j^k)(t) \to 0$$

by the conservation law.

Step 2: Descriptions of the space E

Define

$$\mathcal{N}(T) = \{\mathbb{P}^* V(0) \in \mathcal{K}_0 : (b_1\kappa_1 v_1 + \cdots + b_n\kappa_n v_n)(t,x) = 0, \text{ for } t \in (0,T), x \in \omega\}. \tag{3.4.19}$$

Lemma 3.4.13. *$E = \mathcal{N}(T)$ where E was defined in Equation* (3.4.6) *and E has a finite dimension.*

Proof. According to the weak observability inequality Equation (3.4.9), for $\mathbb{P}^*V(0) \in \mathcal{N}(T)$, we obtain that

$$||\mathbb{P}^*V(0)||_{\mathcal{K}_0}^2 \leq C||\mathbb{P}^*V(0)||_{\mathcal{K}_{-1}}^2. \tag{3.4.20}$$

We know that $\mathcal{N}(T)$ is a closed subspace of $\mathcal{K}_0$. By the compact embedding $\mathcal{K}_0 \hookrightarrow \mathcal{K}_{-1}$, we know that $\mathcal{N}(T)$ has a finite dimension. By definition, we know that $E \subset \mathcal{N}(T)$. Hence, we obtain that E has a finite dimension. Then we want to show that $E = \mathcal{N}(T)$. Define

$$\mathscr{A} = \begin{pmatrix} 0 & -1 & \cdots & 0 \\ -\Delta_{K_1} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & -1 \\ 0 & 0 & -\Delta_{K_n} & 0 \end{pmatrix}.$$

Thus, the solution $(v_1, \partial_t v_1, \cdots, v_n, \partial_t v_n)^t$ can be written as

$$\begin{pmatrix} v_1 \\ \partial_t v_1 \\ \vdots \\ v_n \\ \partial_t v_n \end{pmatrix} = e^{-t\mathscr{A}}\mathbb{P}^*V(0).$$

Since $\mathcal{N}(T)$ is of finite dimension, it is complete for any norm. Setting $\delta > 0$(see Remark 3.4.14), we know that Equation (3.4.20) is still true for $\mathbb{P}^*V(0) \in$

$\mathscr{N}(T-\delta)$. Taking $\mathbb{P}^*V(0) \in \mathscr{N}(T)$, for $\epsilon \in]0,\delta[$, we have $e^{-\epsilon\mathscr{A}}\mathbb{P}^*V(0) \in \mathscr{N}(T-\delta)$. For α large enough, as $\epsilon \to 0^+$,

$$(\alpha+\mathscr{A})^{-1}\frac{1}{\epsilon}(Id - e^{-\epsilon\mathscr{A}})\mathbb{P}^*V(0) \to \mathscr{A}(\alpha+\mathscr{A})^{-1}\mathbb{P}^*V(0),$$

as $(\alpha+\mathscr{A})^{-1}\mathbb{P}^*V(0) \in D(\mathscr{A})$. Hence, we know that $\{\frac{1}{\epsilon}(Id - e^{-\epsilon\mathscr{A}})\mathbb{P}^*V(0)\}_{\epsilon>0}$ is a Cauchy sequence in $\mathscr{N}(T-\delta)$, endowed with the norm $||(\alpha+\mathscr{A})^{-1}\cdot||_{\mathcal{K}_1}$. Since all norms are equivalent, we obtain a Cauchy sequence $\{\frac{1}{\epsilon}(Id - e^{-\epsilon\mathscr{A}})\mathbb{P}^*V(0)\}_{\epsilon>0}$ in $\mathscr{N}(T-\delta)$, endowed with the norm $||\cdot||_{\mathcal{K}_1}$, which yields $\mathscr{A}\mathbb{P}^*V(0) \in \mathcal{K}_1$. As a consequence, we obtain $\mathscr{N}(T) \subset D(\mathscr{A}) \subset \mathcal{K}_1$. Hence, we obtain that $E = \mathscr{N}(T)$ and has a finite dimension. One can see [21] for more details. $\square$

Remark 3.4.14. *One has to take δ small enough. Actually, if T_0 is the constant such that (ω, T_0) satisfies GCC, and $T > T_0$, one is able to choose, for example, $\delta = \frac{T-T_0}{2}$.*

Step 3: Proof of the observability inequality Equation (3.4.8)

If Equation (3.4.8) was false, we could find a sequence $\{\mathbb{P}^*V^k(0)\}_{k\in\mathbb{N}} \subset \mathcal{K}_0$ such that

$$||\mathbb{P}^*V^k(0)||_{\mathcal{K}_0} = 1, \quad \int_0^T ||b_1\kappa_1 v_1^k + \cdots + b_n\kappa_n v_n^k||^2_{L^2(\omega)}dt \to 0. \tag{3.4.21}$$

First, we know that $\{\mathbb{P}^*V^k(0)^k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{K}_0 = (L^2 \times H^{-1})^n$. Hence, there exists a subsequence (also denoted by $\mathbb{P}^*V^k(0)$) weakly converging in $\mathcal{K}_0 = (L^2 \times H^{-1})^n$, to a limit which we denote with $\mathbb{P}^*V(0)$. We also know that $\mathbb{P}^*V(0)$ leads to a solution $(v_1, \cdots, v_n)$ of the system Equation (3.4.7) and satisfies that $b_1\kappa_1v_1 + \cdots + b_n\kappa_nv_n = 0$ in $(0,T)\times\omega$. Thus, by the definition of $\mathscr{N}(T)$ (see Equation (3.4.19)), we know that $\mathbb{P}^*V(0) \in \mathscr{N}(T) = E$, which implies that $\mathbb{P}^*V(0) = 0$. Since the embedding $\mathcal{K}_0 \hookrightarrow \mathcal{K}_{-1}$ is compact, we obtain that $||\mathbb{P}^*V(0)^k||^2_{\mathcal{K}_{-1}} \to ||\mathbb{P}^*V(0)||^2_{\mathcal{K}_{-1}}$. From the weak observability inequality Equation (3.4.9), we obtain:

$$1 \le C||\mathbb{P}^*V(0)||^2_{\mathcal{K}_{-1}},$$

which contradicts to the fact that $\mathbb{P}^*V(0) = 0$. Then observability inequality Equation (3.4.8) follows. This concludes the proof of the $\mathbb{P}-$exact controllability of the system Equation (3.4.4).

3.4.3 The Proof of Theorem 3.2.5

According to the proof above, we only need to show that $E^\perp = \{0\}$, which is equivalent to $\mathbb{P}^* = Id$. If we denote by $\tilde{V}(t)$ the solution of

$$\partial_t\tilde{V} + \mathscr{A}\tilde{V} = 0, \tilde{V}|_{t=0} = V(0),$$

then, $\mathscr{A}V(0) = -\partial_t \tilde{V}|_{t=0} \in \mathscr{N}(T)$ provided that $V(0) \in \mathscr{N}(T)$. This implies that $\mathscr{A}\mathscr{N}(T) \subset \mathscr{N}(T)$. Since $\mathscr{N}(T)$ is a finite dimensional closed subspace of $D(\mathscr{A})$, and stable by the action of the operator $\mathscr{A}$, it contains an eigenfunction of $\mathscr{A}$. To be specific, there exists $(e_1, e_2, \cdots, e_n) \in \mathscr{N}(T)$ and $\lambda \in \mathbb{C}$ such that

$$\begin{pmatrix} 0 & -1 & \cdots & 0 \\ -\Delta_{K_1} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & -1 \\ 0 & 0 & -\Delta_{K_n} & 0 \end{pmatrix} \begin{pmatrix} e_1^0 \\ e_1^1 \\ \vdots \\ e_n^0 \\ e_n^1 \end{pmatrix} = \lambda \begin{pmatrix} e_1^0 \\ e_1^1 \\ \vdots \\ e_n^0 \\ e_n^1 \end{pmatrix}.$$

It is equivalent to the following system:

$$\begin{cases} -e_1^1 = \lambda e_1^0 \text{ in } \Omega, \\ -\Delta_{K_1} e_1^0 = \lambda e_1^1 \text{ in } \Omega, \\ \cdots \\ -e_n^1 = \lambda e_n^0 \text{ in } \Omega, \\ -\Delta_{K_n} e_n^0 = \lambda e_n^1 \text{ in } \Omega, \\ b_1 \kappa_1 e_1^0 + \cdots + b_n \kappa_n e_n^0 = 0, \text{ in } \omega. \end{cases} \tag{3.4.22}$$

We can simplify this into

$$\begin{cases} \Delta_{K_1} e_1^0 = \lambda^2 e_1^0 \text{ in } \Omega, \\ \Delta_{K_2} e_2^0 = \lambda^2 e_2^0 \text{ in } \Omega, \\ \cdots \\ \Delta_{K_n} e_n^0 = \lambda^2 e_n^0 \text{ in } \Omega, \\ b_1 \kappa_1 e_1^0 + \cdots + b_n \kappa_n e_n^0 = 0 \text{ in } \omega, \end{cases}$$

Since the system satisfies the unique continuation of eigenfunctions, we know that $e_1^0 = \cdots = e_n^0 = 0$ in Ω, which implies that $E = \mathcal{N}(T) = \{0\}$. Hence, from Equation (3.4.8) with $\mathbb{P}^* = Id$, we obtain the observability inequality

$$C \int_0^T \int_\omega |b_1 \kappa_1 v_1 + \cdots + b_n \kappa_n v_n|^2 dx dt \geq ||V(0)||_{\mathcal{K}_0}^2.$$

This concludes the proof of Theorem 3.2.5.

3.5 Unique continuation of eigenfunctions

3.5.1 A counterexample

First, we construct an example to show that the conditions in Theorem 3.2.1 are not sufficient to ensure the unique continuation of eigenfunctions. Now, let us focus on the unique continuation problem in dimension 1. We consider a smooth metric in dimension 1, $g = c(x)dx^2$. Then we can define the Laplace-Beltrami operator in the sense:

$$\begin{aligned}\Delta_g &= \frac{1}{\sqrt{\det(g)}}\frac{d}{dx}(\sqrt{\det(g)}g^{-1}\frac{d}{dx}) \\ &= \frac{1}{c}\frac{d^2}{dx^2} - \frac{c'}{2c^2}\frac{d}{dx}\end{aligned} \tag{3.5.1}$$

Fix the open interval $(0, \pi)$ and the subinterval $(a, b) \subset (0, \pi)(a > \frac{\pi}{2})$. Now we consider the unique continuation problem:

$$\begin{cases} u_1'' = -\lambda^2 u_1, \\ \Delta_g u_2 = -\lambda^2 u_2, \\ u_1 + u_2 = 0 \text{ in } (a, b), \\ u_1, u_2 \in H_0^1((0, \pi)). \end{cases} \tag{3.5.2}$$

In general, the unique continuation of eigenfunctions does not hold.

Theorem 3.5.1. *There exists a smooth Riemannian metric $g = c(x)dx^2$, and two eigenfunctions u_1, u_2 of Δ_g and $\frac{d^2}{dx^2}$ on $(0, \pi)$ associated with eigenvalue 1 such that $u_1 + u_2 = 0$, in $(a, b) \subset (0, \pi)$ and $u_1 + u_2 \not\equiv 0$ in $(0, \pi)$.*

Proof. Let $\chi \in C^\infty(\mathbb{R})$ satisfying the following conditions:

1. $\chi(0) = \chi(\pi) = 0$;
2. $0 < \chi \leq K$ on $(0, \pi)$ and $\chi(\frac{\pi}{2}) = K > 1$;
3. $\chi(x) = 1, \forall x \in (a, b)$;
4. $\chi'(x) > 0$ for $x \in [0, \frac{\pi}{2}[$, $\chi'(x) < 0$ for $x \in]b, \pi]$ and $\chi'(x) < 0$ for $x \in]\frac{\pi}{2}, a[$

Define $u_2(x) = -\chi(x)\sin x$. Hence, we obtain $u_2(x) = -\sin x$ on (a, b) and $u_2'(x) = -\chi'(x)\sin x - \chi(x)\cos x$. Then we define $c(x)$ by

$$c(x) = \frac{(\chi'(x)\sin x + \chi(x)\cos x)^2}{K^2 - \chi^2\sin^2 x}, \tag{3.5.3}$$

with a constant $K > 1$. It is easy to check that $c \geq 0$. Since we want g to be a Riemannian metric, we need $c > 0$. Let us discuss in different cases,

1. if $x \in]0, \frac{\pi}{2}[$, we know that $\chi'(x) > 0$, $\chi(x) > 0$. Hence, we have $\chi'(x)\sin x + \chi(x)\cos x > 0$;

2. if $x \in [a,b]$, $\chi'(x) = 0$, $\chi(x) = 1$, we obtain $\chi'(x)\sin x + \chi(x)\cos x = \cos x < 0$ since $a > \frac{\pi}{2}$;

3. if $x \in]b,\pi[$, we know that $\chi'(x) < 0$, $\chi(x) > 0$. Hence, we have $\chi'(x)\sin x + \chi(x)\cos x < 0$;

4. if $x \in]\frac{\pi}{2}, a[$, we know that $\chi'(x) < 0$, $\chi(x) > 0$. Hence, we have $\chi'(x)\sin x + \chi(x)\cos x < 0$;

5. if $x = \frac{\pi}{2}$, $\chi'(\frac{\pi}{2}) = 0$, $c(\frac{\pi}{2}) = 1 - \frac{\chi''(\frac{\pi}{2})}{K} \geq 1$.

So we can conclude that $c > 0$ and g is a Riemannian metric.

We want to show that c is C^∞ near $\frac{\pi}{2}$. Let $f(x) = (\chi'(x)\sin x + \chi(x)\cos x)^2$ and $g(x) = K^2 - \chi^2 \sin^2 x$, then we obtain $c(x) = \frac{f}{g}$. We claim that there exist $\tilde{f}, \tilde{g} \in C^\infty$ and $\tilde{f}(\frac{\pi}{2}) \neq 0$, $\tilde{g}(\frac{\pi}{2}) \neq 0$ such that $f(x) = (x - \frac{\pi}{2})^2 \tilde{f}(x)$ and $g(x) = (x - \frac{\pi}{2})^2 \tilde{g}(x)$. We just use the Taylor expansion of χ, χ', sin and cos:

$$\begin{aligned}
\chi(x) &= K + \frac{1}{2}\chi''(\frac{\pi}{2})(x - \frac{\pi}{2})^2 + R_1(x), \\
\chi'(x) &= \chi''(\frac{\pi}{2})(x - \frac{\pi}{2}) + \frac{1}{2}\chi'''(\frac{\pi}{2})(x - \frac{\pi}{2})^2 + R_2(x), \\
\sin(x) &= 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 + R_3(x), \\
\cos(x) &= -(x - \frac{\pi}{2}) + R_4(x),
\end{aligned} \tag{3.5.4}$$

where $\lim_{x \to \frac{\pi}{2}} \frac{R_j}{(x - \frac{\pi}{2})^2} = 0$, for $j = 1,2,3,4$. Then we obtain:

$$\begin{aligned}
f(x) &= ((\chi''(\frac{\pi}{2}) - K)^2 + \tilde{R}_1)(x - \frac{\pi}{2})^2; \\
g(x) &= (-K(\chi''(\frac{\pi}{2}) - K) + \tilde{R}_2)(x - \frac{\pi}{2})^2.
\end{aligned} \tag{3.5.5}$$

Here $\lim_{x \to \frac{\pi}{2}} \tilde{R}_j = 0$ for $j = 1,2$. Now if we choose a small neighbourhood of $\frac{\pi}{2}$, then $\tilde{f} = (\chi''(\frac{\pi}{2}) - K)^2 + \tilde{R}_1$ and $\tilde{g} = -K(\chi''(\frac{\pi}{2}) - K) + \tilde{R}_2$ satisfy the property. So we know c is C^∞ and $c > 0$, which means that g is a smooth Riemannian metric. In addition, $c < 1$ in (a,b) and Δ_g and Δ admit the same eigenfunction in this interval (a,b). □

Remark 3.5.2. *In fact, we can construct a counterexample in any dimension $d \geq 1$. For example, we define $M = (0,\pi) \times \Pi_y^{d-1}$ where Π_y^{d-1} is the torus of dimension $d-1$. Then consider two metric $g_1 = \mathrm{d}x^2 + \sum_{j=0}^{d-1} \mathrm{d}y_j^2$ and $g_2 = c(x)\,\mathrm{d}x^2 + \sum_{j=0}^{d-1} \mathrm{d}y_j^2$ where $c(x)\,\mathrm{d}x^2$ is the metric we constructed in the dimension 1. Take the same*

$u_1(x)$ and $u_2(x)$ in the proof of Theorem 3.5.1. Let V be the eigenfunction of $\sum_{j=1}^{d-1}\frac{\mathrm{d}^2}{\mathrm{d}y_j^2}$ associated with eigenvalue α in Π_y^{d-1}. Then

$$\begin{cases} -\Delta_{g_1}(u_1(x)V(y)) = (\alpha+1)u_1(x)V(y), \\ -\Delta_{g_2}(u_2(x)V(y)) = (\alpha+1)u_2(x)V(y), \\ u_1(x)V(y) + u_2(x)V(y) = 0, \ \textit{in } (a,b)\times\Pi_y^{d-1}, \\ u_1(x)V(y), u_2(x)V(y) \in H_0^1(M). \end{cases}$$

But we know $u_1(x)V(y) + u_2(x)V(y) \not\equiv 0$ in M.

As we have seen, not every smooth metric can give us the unique continuation of eigenfunctions. Here, we will give a positive result under a strong condition of analyticity. In particular, let us consider the example of two equations:

$$\begin{cases} \Box_{K_1}u_1 = b_1 f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T)\times\Omega \\ \Box_{K_2}u_2 = b_2 f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T)\times\Omega \\ u_j = 0 \quad \text{on } (0,T)\times\partial\Omega, j=1,2, \\ u_j(0,x) = u_j^0(x), \quad \partial_t u_j(0,x) = u_j^1(x), j=1,2. \end{cases} \tag{3.5.6}$$

Proposition 3.5.3. *Given $T>0$, suppose that:*

1. *(ω, T, p_{K_i}) satisfies GCC, $i=1,2$.*

2. *$K_1 > K_2$ in Ω with analytic coefficients.*

3. *There exists a constant c such that density functions κ_1, κ_2 are analytic and $\kappa_1 = c\kappa_2$.*

4. *Ω has no infinite order of contact on the boundary.*

Then the system Equation (3.5.6) is exactly controllable.

Proof. According to Theorem 3.2.1, we only need to show the unique continuation of eigenfunctions of system Equation (3.5.6):

$$\begin{cases} -\Delta_{K_1}u_1 = \lambda^2 u_1 \text{ in } \Omega, \\ -\Delta_{K_2}u_2 = \lambda^2 u_2 \text{ in } \Omega, \\ cu_1 + u_2 = 0 \text{ in } \omega. \end{cases} \tag{3.5.7}$$

Since K_1 and K_2 have analytic coefficients, we know u_1 and u_2 are analytic functions. Then cu_1+u_2 is also analytic. By unique continuation for analytic functions,

$cu_1 + u_2 = 0$ in the whole domain Ω. By the relations of two density functions $\kappa_1 = c\kappa_2$, we have:

$$\begin{aligned}\Delta_{K_1} u_1 &= \frac{1}{\kappa_1(x)} div(\kappa_1(x) K_1 \nabla u_1) \\ &= \frac{1}{c\kappa_2(x)} div(c\kappa_2(x) K_1 \nabla u_1) \\ &= \frac{1}{\kappa_2(x)} div(\kappa_2(x) K_1 \nabla u_1).\end{aligned} \tag{3.5.8}$$

Then

$$\begin{aligned}-c\Delta_{K_1} u_1 - \Delta_{K_2} u_2 &= -\frac{c}{\kappa_2(x)} div(\kappa_2(x) K_1 \nabla u_1) - \frac{1}{\kappa_2(x)} div(\kappa_2(x) K_2 \nabla u_2) \\ &= -\frac{c}{\kappa_2(x)} div(\kappa_2(x) K_1 \nabla u_1) + \frac{c}{\kappa_2(x)} div(\kappa_2(x) K_2 \nabla u_1) \\ &= -\frac{c}{\kappa_2(x)} div(\kappa_2(x)(K_1 - K_2) \nabla u_2).\end{aligned}$$

On the other hand, we know $-c\Delta_{K_1} u_1 - \Delta_{K_2} u_2 = \lambda^2(cu_1 + u_2) = 0$. Hence, we have:

$$-\frac{1}{\kappa_2(x)} div(\kappa_2(x)(K_1 - K_2) \nabla u_1) = 0.$$

We recall that $-\frac{1}{\kappa_2(x)} div(\kappa_2(x)(K_1 - K_2)\nabla \cdot)$ is an elliptic operator. Hence, with $u_1|_{\partial\Omega} = 0$ on the boundary, we know that $u_1 = 0$. Hence, we deduce $u_2 = -cu_1 = 0$ in Ω, which gives $\mathcal{N}(T) = 0$. □

3.5.2 Constant Coefficient Case

In this section, we consider the simultaneous control problem for the system:

$$\partial_t^2 U - D\Delta U = Bf \mathbf{1}_{(0,T)}(t) \mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \tag{3.5.9}$$

where $U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ and $D = diag(d_1, \cdots, d_n)$. Then the system can be written as

$$\begin{cases} (\partial_t^2 - d_1\Delta) u_1 = b_1 f \mathbf{1}_{(0,T)}(t) \mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ \vdots \\ (\partial_t^2 - d_n\Delta) u_n = b_n f \mathbf{1}_{(0,T)}(t) \mathbf{1}_\omega(x) \text{ in } (0,T) \times \Omega, \\ u_j = 0 \quad \text{on } (0,T) \times \partial\Omega, 1 \le j \le n, \\ u_j(0,x) = u_j^0(x), \quad \partial_t u_j(0,x) = u_j^1(x), 1 \le j \le n. \end{cases}$$

Recall that the Kalman rank condition for this case is $rank[D|B] = n$ if and only if all d_j are distinct and $b_j \neq 0$, $1 \leq j \leq n$(See [6]). Without loss of generality, we may assume that $d_1 < d_2 < \cdots < d_n$. We want to prove the exact controllability for this case(Theorem 3.2.8).

3.5.3 Proof of Theorem 3.2.8

By Theorem 3.2.1, we only need to prove the unique continuation properties for eigenfunctions. Here we only state some facts without repeating the same trick as before. Define

$$\mathscr{N}(T) = \{\mathscr{V} \in (L^2 \times H^{-1})^n : (b_1 v_1 + b_2 v_2 + \cdots + b_n v_n)(x,t) = 0, \forall (x,t) \in (0,T) \times \omega\}.$$

Then, $\mathscr{N}(T)$ is a finite dimensional closed subspace of $D(\mathscr{A})$, and stable by the action of the operator $\mathscr{A}$, it contains an eigenfunction of $\mathscr{A}$, where $\mathscr{A} = \begin{pmatrix} 0 & -Id \\ -D\Delta & 0 \end{pmatrix}$. Thus there exist $\beta \in \mathbb{C}$ and $\mathscr{V}_\beta = (V_1, V_2)$ such that $\mathscr{A}\mathscr{V}_\beta = \beta \mathscr{V}_\beta$, i.e.

$$-\Delta V_1 = -\beta^2 D^{-1} V_1 \tag{3.5.10}$$

If $\beta \neq 0$, $(-\beta^2)^{-k}(-\Delta)^k V_1 = D^{-k} V_1$ and $(-\Delta)^k B^t V_1 = (-\beta^2)^k B^t D^{-k} V_1$. Since V_1 solves the Laplace eigenvalue problem, we know that V_1 is analytic in Ω which ensures that $B^t V_1 = b_1 v_1^1 + \cdots + b_n v_1^n = 0$ in the whole domain Ω. Thus

$$0 = [B^t V_1 | (-\beta^2)^{-1}(-\Delta) B^t V_1 | \cdots | (-\beta^2)^{-n}(-\Delta)^n B^t V_1] = [D|B]^t D^{1-n} V_1 \tag{3.5.11}$$

Since $rank[D|B] = n$, it is invertible. This gives that $V_1 = 0$.

If $\beta = 0$, we immediately obtian that $V_1 = 0$ by the boundary condition.

Now we assume that the matrix (D, B) does not satisfy the Kalman rank condition. Then we know that either there exist d_{j_1} and d_{j_2} such that $d_{j_1} = d_{j_2}$, or there exists some $b_j = 0$. We want to show the unique continuation property fails in both cases. One can refer to [22] for more details.

For the first case $b_j = 0$, we know that

$$(\partial_t^2 - d_j \Delta) u_j = 0 \text{ in } (0,T) \times \Omega,$$

by the conservation of energy, the solution u_j cannot be zero at any time if the initial data is not zero.

For the second case, we consider the unique continuation property of the eigenfunctions as follows:

$$\begin{cases} -d_1\Delta\phi_1 = \lambda^2\phi_1 \text{ in } \Omega, \\ \vdots \\ -d_{j_1}\Delta\phi_{j_1} = \lambda^2\phi_{j_1} \text{ in } \Omega, \\ -d_{j_2}\Delta\phi_{j_2} = \lambda^2\phi_{j_2} \text{ in } \Omega, \\ \vdots \\ -d_n\Delta\phi_n = \lambda^2\phi_n \text{ in } \Omega, \\ \phi_j = 0 \quad \text{on } \partial\Omega, 1 \le j \le n, \\ b_1\phi_1 + \cdots + b_n\phi_n = 0 \text{ in } \omega, \end{cases}$$

Since we have the relation $d_{j_1} = d_{j_2}$, we know that there exists a non-zero solution $(0, \cdots, 0, \phi, -\frac{b_{j_1}}{b_{j_2}}\phi, 0, \cdots, 0)$, where ϕ is an eigenfunction for $-d_{j_1}\Delta$ of eigenvalue λ^2. Hence, we cannot obtain the exact controllability in this case.

To conclude, we have obtained that the Kalman rank condition is a sufficient and necessary condition for the exact controllabilty.

3.5.4 Two Generic Properties

If we define $\Delta_{K_1} = \Delta = \frac{d^2}{dx^2}$ and $n = 2$, we have shown that not every smooth metric can give us a unique continuation result in dimension 1 (see Subsection 3.5.1). Then we want to prove a generic property for the metrics which can give the unique continuation result in dimension 1. We introduce the following space of smooth metrics to be sections of a bundle endowed with $C^\infty-$topology

$$\mathcal{M} = \{g \in C^\infty(\Omega, T^*\Omega \otimes T^*\Omega) : g(x)(v_x, v_x) > 0, \text{ for } 0 \neq v_x \in T_x\Omega\}.$$

Let $\Omega = (0, \pi)$.

Proposition 3.5.4. *In dimension* 1*, suppose that we fix the Laplacian* $\Delta = \frac{d^2}{dx^2}$ *in* $(0, \pi)$ *with its spectrum* $\sigma(\Delta)$*. Then the set* $\mathcal{G}_{uc} = \{g \in \mathcal{M} : \sigma(\Delta_g) \cap \sigma(\Delta) = \emptyset\}$ *is residual in* $\mathcal{M}$*.*

Proof. First, we notice that any connected one dimensional Riemannian manifold is diffeomorphic either to $\mathbb{R}$ or to S^1. We already know that $\sigma(\Delta) = \{k^2\}_{k\in\mathbb{N}}$. In our setting, we have $g = c(x)dx^2$. Then by change of variables, $y = \int_0^x \sqrt{c(s)}ds$. Then $\frac{d}{dy} = \frac{dx}{dy}\frac{d}{dx} = \frac{1}{\sqrt{c(x)}}\frac{d}{dx}$. Hence, we obtain $\frac{d^2}{dy^2} = \frac{1}{\sqrt{c(x)}}\frac{d}{dx}\frac{1}{\sqrt{c(x)}}\frac{d}{dx} = \Delta_g$. Define $L = \int_0^\pi \sqrt{c(s)}ds$. Hence, $\sigma(\Delta_g) = \sigma(\frac{d^2}{dy^2}) = \{\frac{k^2\pi^2}{L^2}\}_{k\in\mathbb{N}}$. If $\sigma(\Delta_g) \cap \sigma(\Delta) \neq \emptyset$, we obtain that for some k and l, $L = \frac{k\pi}{l} \in \pi\mathbb{Q}$, i.e. $\int_0^\pi \sqrt{c(x)}\,\mathrm{d}x \in \pi\mathbb{Q}$. □

Corollary 3.5.5. *Fix* $\Delta = \frac{d^2}{dx^2}$*, for every metric* $g \in \mathcal{G}_{uc}$*, the system Equation* (3.5.2) *has a unique solution* $u_1 = u_2 = 0$*.*

Proof. By the definition of $\mathcal{G}_{uc}$, we know $\sigma(\Delta_g) \cap \sigma(\Delta) = \emptyset$. Consider a solution u_1, u_2 of

$$\begin{cases} u_1'' = -\lambda^2 u_1, \\ \Delta_g u_2 = -\lambda^2 u_2, \\ u_1 + u_2 = 0 \text{ in } (a,b), \\ u_1, u_2 \in H_0^1((0,\pi)). \end{cases}$$

Now, assume that $u_1 = 0$. Then $u_2 = 0$ in (a,b). Hence, by the unique continuation property for the eigenfunctions, we know that $u_2 = 0$. This means that the system has only trivial solution in this case. It is the same for $u_2 = 0$.

Assume that $u_1 \neq 0$ then $u_1 \neq 0$ in (a,b)(otherwise $u_1 = 0$ everywhere by the unique continuation property) and therefore $u_2 \neq 0$. Then u_1 and u_2 are both eigenfunctions. Hence $\lambda^2 \in \sigma(\Delta_g) \cap \sigma(\Delta) = \emptyset$, which is a contradiction. So for every $g \in \mathcal{G}_{uc}$, the system has only the trivial solution $(0,0)$. □

From now on and until the end of the section, we restrict to the 2 dimensional case $d = 2$. For any smooth metric g, we can define a Laplace-Beltrami operator $-\Delta_g$.

Definition 3.5.6. *Define the map:*

$$\mathcal{E}^\alpha : H^2(\Omega) \cap H_0^1(\Omega) \times \mathcal{M} \to L^2$$

by $\mathcal{E}^\alpha(u,g) = (\Delta_g + \alpha)u$.

Remark 3.5.7. *$-\Delta_g$ is a Fredholm operator of index 0, and $\mathcal{E}_g^\alpha = \mathcal{E}^\alpha(\cdot, g)$ is also a Fredholm map of index 0(see [51]). Here α is just a parameter. In the later proof, we will let α take all possible values in the spectrum of the given Laplacian.*

From now on, we fix one metric g_0 and the associted operator $-\Delta_{g_0}$.

Lemma 3.5.8. *For any λ fixed and any element $f \in L^2$, $\lambda \notin \sigma(\Delta_g)$ if and only if f is a regular value (i.e. the tangential map at this point is surjective) of $\mathcal{E}_g^\lambda : H^2(\Omega) \cap H_0^1(\Omega) \to H^{-1}$.*

Proof. Let $\mathcal{E}_g^\lambda(u) = \mathcal{E}^\lambda(u,g) = f$. At this point u, the tangential map $D\mathcal{E}_g^\lambda : T_u(H^k(\Omega) \cap H_0^1(\Omega)) \to H^{-1}(\Omega)$ is given by $D\mathcal{E}_g^\lambda(v) = (\Delta_g + \lambda)v$, since $\Delta_g + \lambda$ is a linear operator. $\lambda \notin \sigma(\Delta_g)$ is equivalent to that $\Delta_g + \lambda$ is bijective, which means f is a regular value of $\mathcal{E}_g^\lambda$. □

Our proof mainly relies on the following theorem:

Theorem 3.5.9 (Transversality theorem)**.** *Let $\varphi : H \times B \to E$ be a C^k map, H, B, and E Banach manifolds with H and E separable. If f is a regular value of φ and $\varphi_b = \varphi(\cdot, b)$ is a Fredholm map of index $< k$, then the set $\{b \in B : f \text{ is a regular value of } \varphi_b\}$ is residual in B.*

One can find a proof in [1].

Lemma 3.5.10. *If $\lambda \in \sigma(\Delta_{g_0})$ is a regular value of $\mathcal{E}^\lambda$, then the set $\{g \in \mathcal{M} : \lambda \notin \sigma(\Delta_g)\}$ is residual in $\mathcal{M}$.*

Proof. Just apply Theorem 3.5.9, combining with Lemma 3.5.8. □

Now we have to check with the hypothesis, that is to verify that $\lambda \in \sigma(-\Delta_{g_0})$ is a regular value for $\mathcal{E}^\lambda$. In the following, we will use D_1 to denote the differential in the direction of $H^2(\Omega) \cap H_0^1(\Omega)$ and D_2 to denote the differential in the direction of $\mathcal{M}$.

Now let us check that the image of $D_2\mathcal{E}^\lambda$ is dense in dimension 2. We will use the conformal variations of the metric g. Here we choose $r \in C_0^\infty(\Omega)$

$$\begin{aligned} D_2\mathcal{E}^\lambda(rg) &= \lim_{s\to 0} \frac{(\Delta_{g+srg} - \Delta_g)u}{s} \\ &= \lim_{s\to 0} \frac{1}{s}\left(\frac{1}{|(1+sr)g|^{\frac{1}{2}}}\partial_i |(1+sr)g|^{\frac{1}{2}}(1+sr)^{-1}g^{ij}\partial_j u - \Delta_g u\right) \\ &= \lim_{s\to 0} \frac{1}{s}\left(\frac{2-2}{2}(1+sr)^{-2}\partial_i r g^{ij}\partial_j u + \frac{1}{1+sr}\Delta_g u - \Delta_g u\right) \\ &= -r\Delta_g u \end{aligned} \tag{3.5.12}$$

Let us assume that v is orthogonal to $D_2\mathcal{E}^\lambda(rg)$ for all r, then:

$$\begin{aligned} 0 &= \int_\Omega v D_2\mathcal{E}^\lambda(rg)d\mu_g \\ &= \int_\Omega v(-r\Delta_g u)d\mu_g \\ &= \int_\Omega r(\lambda u - \lambda)v d\mu_g. \end{aligned} \tag{3.5.13}$$

Since Equation (3.5.13) holds for any $r \in C_0^\infty(\Omega)$ we obtain that:

$$(\lambda u - \lambda)v = 0. \tag{3.5.14}$$

Now, we can check that λ is a regular value of $\mathcal{E}^\lambda$.

Lemma 3.5.11. *In dimension 2, $\lambda \in \sigma(\Delta_{g_0})$ is a regular value of $\mathcal{E}^\lambda$.*

Proof. Let (u, g) satisfy $\mathcal{E}^\lambda(u, g) = (\Delta_g + \lambda)u = \lambda$, then at the point (u, g), we have

$$D\mathcal{E}^\lambda(v, h) = (\Delta_g + \lambda)v + D_2\mathcal{E}^\lambda(h).$$

Now we need to verify the surjectivity of this map. If $y \in Im(\Delta_g + \lambda)^{\perp}$, then y is a weak solution of $(\Delta_g + \lambda)y = 0$, and y is smooth. Let us assume that y is orthogonal to $D_2\mathcal{E}^{\lambda}(rg)$. Then according to Equation (3.5.14), we obtain that:

$$(\lambda u - \lambda)y = 0.$$

First, we claim that u cannot be a constant. Assume that u is a constant function, $\Delta_g u = 0$ and $(\Delta_g + \lambda)u = \lambda$ gives that $u = 1$. But this does not satisfy the boundary condition. Hence, u cannot be a constant. In particular, $u \not\equiv 1$. Now we obtain that $\lambda u - \lambda \not\equiv 0$. If $\lambda u - \lambda \neq 0$ at x_0, there exists a open neighbourhood N such that $\lambda u - \lambda \neq 0$ in N. Then $y \equiv 0$ in N. Hence, we know that y vanishes in a subdomain of Ω. Then by the unique continuation property, we know $y = 0$ in Ω. This leads to the surjectivity of the map $D\mathcal{E}^{\lambda}$, which means that $\lambda \in \sigma(-\Delta_{g_0})$ is a regular value of $\mathcal{E}^{\lambda}$. □

Now we can deduce that the set $G^{\lambda} = \{g \in \mathcal{M} : \lambda \notin \sigma(\Delta_g)\}$ is residual in $\mathcal{M}$.

Proposition 3.5.12. *In dimension* 2*, suppose that we fix one metric* g_0 *and the associated Laplacian* Δ_{g_0} *with its spectrum* $\sigma(\Delta_{g_0})$*. Then the set* $\mathcal{G}_{uc} = \{g \in \mathcal{M} : \sigma(\Delta_g) \cap \sigma(\Delta_{g_0}) = \emptyset\}$ *is residual in* $\mathcal{M}$*.*

Proof. Define:

$$\mathcal{G}_{uc} = \cap_{\lambda \in \sigma(\Delta_{g_0})} G^{\lambda}.$$

G is a intersection of countably many residual sets, so it is still residual in $\mathcal{M}$. And for any metric $g \in \mathcal{G}_{uc}$, $\sigma(\Delta_g) \cap \sigma(\Delta_{g_0}) = \emptyset$. Assume that $\lambda_0 \in \sigma(\Delta_g) \cap \sigma(\Delta_{g_0})$, which gives that $g \notin G^{\lambda_0}$. That contradicts to the fact that $g \in \mathcal{G}_{uc} = \cap_{\lambda \in \sigma(\Delta)} G^{\lambda}$. Hence, for fixed Laplacian Δ with its spectrum $\sigma(\Delta_{g_0})$, the set $\{g \in \mathcal{M} : \sigma(\Delta_g) \cap \sigma(\Delta_{g_0}) = \emptyset\}$ is residual in $\mathcal{M}$. □

Corollary 3.5.13. *In dimension* 2*, fix the canonical Laplace operator* Δ*, for every metric* $g \in \mathcal{G}_{uc}$*, the system*

$$\begin{cases} \Delta u_1 = -\lambda^2 u_1, \\ \Delta_g u_2 = -\lambda^2 u_2, \\ u_1 + u_2 = 0 \ \textit{in} \ \omega \subset \Omega, \\ u_1, u_2 \in H_0^1(\Omega), \end{cases}$$

has only trivial solution $u_1 = u_2 = 0$*.*

3.6 Constant Coefficient Case with Multiple Control Functions

In this section, we prove Theorem 3.2.10. First we study the information given by the Kalman rank condition. Without loss of generality, we assume that the diagonal matrix D has the form $D = \begin{pmatrix} d_1 Id_{n_1} & & \\ & \ddots & \\ & & d_s Id_{n_s} \end{pmatrix}$, where $\sum_{1\leq i\leq s} n_i = n$ and $d_i (1 \leq i \leq s)$ are all distinct. And we can always rearrange the lines of the system Equation (3.2.2) to ensure that this property is verified:

$$\begin{cases} (\partial_t^2 - d_1\Delta)U_1 = B_1 F \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T)\times\Omega, \\ \qquad\vdots \\ (\partial_t^2 - d_s\Delta)U_s = B_s F \mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \text{ in } (0,T)\times\Omega, \end{cases}$$

for every $1 \leq i \leq s$, where $U_i = \begin{pmatrix} u_1^i \\ \vdots \\ u_{n_i}^i \end{pmatrix}$ and $B_i = \begin{pmatrix} b_{11}^i & \cdots & b_{1m}^i \\ \vdots & \ddots & \vdots \\ b_{n_i 1}^i & \cdots & b_{n_i m}^i \end{pmatrix}$ is a matrix of size $n_i \times m$.

Proposition 3.6.1. *(D, B) satisfies the Kalman rank condition if and only if $rank(B_i) = n_i \leq m$.*

Remark 3.6.2. *If $m = 1$, we know that $rank(B_i) = n_i \leq 1$. Thus, we obtain $n_i = 1$ and $B_i = b_i \neq 0$. This implies that every entry of control matrix B is nonzero and all speeds d_i are distinct. We recover the result of Remark 1.1 in [6]. If $m \geq 2$, we can allow some block $d_i \mathrm{Id}_{n_i}$ is of size $n_i \times n_i$, with $n_i \geq 2$. For example, take $D = diag(1,1,2)$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we obtain $[D|B] = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 4 & 0 & 2 & 0 & 1 & 0 \end{pmatrix}$. Hence, we know that $rank[D|B] = 3$ which means that the matrix $[D|B]$ has full rank.*

The proof of Proposition 3.6.1 is given in the Appendix.

Now we can prove Theorem 3.2.10.

Proof of Theorem 3.2.10. We follow the same procedure. Applying Hilbert uniqueness method, we can estabish the observability inequality:

$$||V(0)||^2_{(L^2\times H^{-1})^n} \leq C \int_0^T \int_\omega |B^*V|^2 dxdt, \tag{3.6.1}$$

where B^* is the adjoint form of the matrix B, and $V = (V_1, \cdots, V_s)^t \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_s} = \mathbb{R}^n$. Then we can estabilsh a similar weak observability inequality:

$$||V(0)||^2_{(L^2\times H^{-1})^n} \leq C \int_0^T \int_\omega |B^*V|^2 dxdt + C||V(0)||^2_{(H^{-1}\times H^{-2})^n}. \tag{3.6.2}$$

Then argue by contradiction. Suppose that the weak observability inequality is false, then there exists a sequence $(V^k(0))_{k\in\mathbb{N}}$ such that

$$||V^k(0)||^2_{(L^2\times H^{-1})^n} = 1, \tag{3.6.3}$$

$$\int_0^T \int_\omega |B^* V^k|^2 dxdt \to 0, \tag{3.6.4}$$

$$||V^k(0)||^2_{(H^{-1}\times H^{-2})^n} \to 0. \tag{3.6.5}$$

Hence, there are s microlocal defect measures $(\mu_i)_{i=1}^s$ corresponding to V_i.

$$\int_0^T \int_\omega |B^* V^k|^2 dxdt = \int_0^T \int_\omega |\sum_{i=1}^s B_i^* V_i^k|^2 dxdt. \tag{3.6.6}$$

Since μ_i and μ_j are singular from each other, for $i \neq j$, we know by Cauchy-Schwarz inequality,

$$\sum_{i=1}^s \int_0^T \int_\omega |B_i^* V_i^k|^2 dxdt \to 0, \tag{3.6.7}$$

which gives that $B_i B_i^* \mu_i|_{\omega\times(0,T)} = 0$. Since $rank(B_i B_i^*) = rank(B_i) = n_i$, we know $B_i B_i^*$ is invertible. Hence we know $\mu_i|_{\omega\times(0,T)} = 0$. The rest of the proof is similar to the single control case.

□

3.7 Appendix I: Proof of Proposition 3.6.1

Proof of Proposition 3.6.1. First, we calculate the form of $[D|B]$:

$$\begin{aligned}[D|B] =& [D^{n-1}B|\cdots|DB|B] \\ =& \begin{bmatrix} d_1^{n-1}B_1 & \cdots & B_1 \\ \vdots & \ddots & \vdots \\ d_s^{n-1}B_s & \cdots & B_s \end{bmatrix}\end{aligned}$$

Now we define $r_i = rank(B_i)$. Thus, for each i, we can find invertible matrices P_i of size $n_i \times n_i$ and Q_i of size $m \times m$ such that $P_i B_i Q_i = \begin{pmatrix} \mathrm{Id}_{r_i} & 0 \\ 0 & 0 \end{pmatrix} \overset{def}{=} E_i$. Then define $P = diag(P_1, \cdots, P_s)$ and $Q = diag(Q_1, \cdots, Q_s)$. We know that P and Q are invertible. Hence, we obtain $rank[D|B] = rank(P[D|B]Q)$. Now we rewrite that

$$P[D|B]Q = \begin{bmatrix} d_1^{n-1}P_1B_1Q_1 & \cdots & P_1B_1Q_s \\ \vdots & \ddots & \vdots \\ d_s^{n-1}P_sB_sQ_1 & \cdots & P_sB_sQ_s \end{bmatrix}$$
$$= \begin{bmatrix} d_1^{n-1}E_1 & \cdots & P_1B_1Q_s \\ \vdots & \ddots & \vdots \\ d_s^{n-1}P_sB_sQ_1 & \cdots & E_s \end{bmatrix}$$

Now, consider the general term $P_iB_iQ_j$:

$$P_iB_iQ_j = P_iB_iQ_iQ_i^{-1}Q_j = E_iQ_i^{-1}Q_j.$$

Hence,

$$P[D|B]Q = \begin{bmatrix} d_1^{n-1}E_1 & \cdots & E_1Q_1^{-1}Q_s \\ \vdots & \ddots & \vdots \\ d_s^{n-1}E_sQ_s^{-1}Q_1 & \cdots & E_s \end{bmatrix}$$

Now we define the column transform T_1:

$$T_1 = \begin{bmatrix} \mathrm{Id}_{n_1} & -\frac{1}{d_1}Q_1^{-1}Q_2 & \cdots & -\frac{1}{d_1^{n-1}}Q_1^{-1}Q_s \\ 0 & \mathrm{Id}_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{Id}_{n_s} \end{bmatrix}$$

It is easy to see that T_1 is invertible and $rank(P[D|B]Q) = rank(P[D|B]QT_1)$.

$$P[D|B]QT_1$$
$$= \begin{bmatrix} d_1^{n-1}E_1 & 0 & \cdots & 0 \\ d_2^{n-1}E_2Q_2^{-1}Q_1 & (\frac{d_2^{n-1}}{d_2} - \frac{d_2^{n-1}}{d_1})E_2 & \cdots & (\frac{d_2^{n-1}}{d_2^{n-1}} - \frac{d_2^{n-1}}{d_1^{n-1}})E_2Q_2^{-1}Q_s \\ \vdots & \vdots & \ddots & \vdots \\ d_s^{n-1}E_sQ_s^{-1}Q_1 & \cdots & \cdots & (\frac{d_s^{n-1}}{d_s^{n-1}} - \frac{d_s^{n-1}}{d_1^{n-1}})E_s \end{bmatrix}.$$

Step by step, we can do the Gaussian elimination and find an invertible matrix T such that:

$$P[D|B]QT = \begin{bmatrix} d_1^{n-1}E_1 & 0 & \cdots & 0 \\ * & d_2^{n-1}(\frac{1}{d_2} - \frac{1}{d_1})E_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & d_s^{n-1}\prod_{i=1}^{s-1}(\frac{1}{d_s} - \frac{1}{d_i})E_s \end{bmatrix}.$$

Then $rank[D|B] = rank(P[D|B]Q) = rank(P[D|B]Q) = \sum_{i=1}^{s} r_i \leq \sum_{i=1}^{s} n_i$. Hence, $n = rank[D|B] = \sum_{i=1}^{s} r_i \leq \sum_{i=1}^{s} n_i = n$. This implies that $rank[D|B] = n \iff \forall i, r_i = n_i$. $\square$

3.8 Appendix II: Extension of Proposition 3.5.12

This section is based on the proof given by Romain Joly. The author would express the sincere gratitude to him for his valuable advice and detailed suggestions. In this section, we would like to remove the dimension restrictions in the Proposition 3.5.12.

Proposition 3.8.1. *Suppose that we fix one metric g_0 and the associated Laplacian Δ_{g_0} with its spectrum $\sigma(\Delta_{g_0})$. Then the set $\mathcal{G}_{uc} = \{g \in \mathcal{M} : \sigma(\Delta_g) \cap \sigma(\Delta_{g_0}) = \emptyset\}$ is residual in $\mathcal{M}$.*

Proof. As usual, we apply the Theorem 3.5.9. We identify the metric space $\mathcal{G}$ with the space of all symmetric positive definite matrices. As we present in section 3.5.4, we define the map $\mathcal{E}^\lambda : H^2(\Omega) \cap H_0^1(\Omega)\backslash\{0\} \times \mathcal{M} \to L^2$. Now we only need to check that 0 is a regular value for $\mathcal{E}^\lambda$. In the following, we will use D_1 to denote the differential in the direction of $H^2(\Omega) \cap H_0^1(\Omega)$ and D_2 to denote the differential in the direction of $\mathcal{M}$.

Now let us check that the image of $D_2\mathcal{E}^\lambda$ is dense in dimension 2. We will use the conformal variations of the metric g. Here we choose $r \in C_0^\infty(\Omega)$(similarly sa we presented in section 3.5.4)

$$D_2\mathcal{E}^\lambda(rg) = -dr\Delta_g u + (d-2)\mathrm{div}(\mathrm{r}\nabla_{\mathrm{g}}\mathrm{u}). \tag{3.8.1}$$

Since we have $\Delta u = -\lambda u$, we obtain that $D_2\mathcal{E}^\lambda(rg) = dr\lambda u + (d-2)\mathrm{div}(\mathrm{r}\nabla_{\mathrm{g}}\mathrm{u})$. Let us assume that v is orthogonal to $D_2\mathcal{E}^\lambda(rg)$ for all r, then:

$$\begin{aligned} 0 &= \int_\Omega vD_2\mathcal{E}^\lambda(rg)d\mu_g \\ &= \int_\Omega v(-dr\lambda u + (d-2)\mathrm{div}(\mathrm{r}\nabla_{\mathrm{g}}\mathrm{u}))\mathrm{d}\mu_{\mathrm{g}} \\ &= -\int_\Omega r(d\lambda uv + (d-2)\nabla_g v \cdot \nabla_g u)d\mu_g. \end{aligned} \tag{3.8.2}$$

Therefore, we obtain that $d\lambda uv + (d-2)\nabla_g v \cdot \nabla_g u = 0$. Since $u \neq 0$, we obtain that the normal derivative of u cannot be identically 0 on the entire boundary. Suppose that at $x_0 \in \partial\Omega$, $\nabla_g u|_\Omega(x_0) \neq 0$. Let $\alpha(t)$ be the integral curve for the field $\nabla_g u$ passing through x_0. Then the equation becomes the ODE(d>2):

$$d\lambda u(\alpha(t))v(\alpha(t)) + (d-2)\frac{d(v(\alpha(t)))}{dt} = 0.$$

Combining with the Dirichlet boundary condition for v, we obatin that $v \equiv 0$, which implies that 0 is a regular value of $\mathcal{E}^\lambda$. $\square$

Chapter 4

Controllability of a coupled wave system with a single control and different speeds

4.1 Introduction and Main Results

4.1.1 General setting

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, be a bounded and smooth domain. We use Δ to denote the canonical Laplace operator on Ω, and Δ_D to denote the Laplace operator with domain $H^2(\Omega) \cap H_0^1(\Omega)$. Let $\Box_1 = \partial_t^2 - d_1\Delta$ and $\Box_2 = \partial_t^2 - d_2\Delta$ be two d'Alembert operators with different constant speeds $d_1 \neq d_2$. Let n_1, n_2 be two integers and $n = n_1 + n_2$. We assume that ω is a nonempty open subset of Ω and that $T > 0$ is a final time horizon. In this article, we aim to deal with some controllability properties of the following type of coupled wave systems:

$$\begin{cases} \Box_1 U_1 + A_1 U_2 & = 0 & \text{in } (0,T) \times \Omega, \\ \Box_2 U_2 + A_2 U_2 & = b f \mathbb{1}_\omega & \text{in } (0,T) \times \Omega, \\ U_1 = U_2 & = 0 & \text{on } (0,T) \times \partial\Omega, \\ (U_1, U_2)|_{t=0} & = (U_1^0, U_2^0) & \text{in } \Omega, \\ (\partial_t U_1, \partial_t U_2)|_{t=0} & = (U_1^1, U_2^1) & \text{in } \Omega. \end{cases} \tag{4.1.1}$$

For $j = 1, 2$, we use $U_j = \begin{pmatrix} u_1^j \\ \vdots \\ u_{n_j}^j \end{pmatrix}$ to denote the solutions corresponding to the speed d_j. $f \in L^2((0,T) \times \omega)$ is the control function, which is a scalar control and acts on $(0,T) \times \omega$. $A_1 \in \mathcal{M}_{n_1,n_2}(\mathbb{R})$ and $A_2 \in \mathcal{M}_{n_2}(\mathbb{R})$ are two given coupling matrices and $b \in \mathbb{R}^{n_2}$. Note that System (4.1.1) is a particular case of systems of the form

$$\begin{cases} (\partial_t^2 - D\Delta)U + AU &= \hat{b} f \mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ U &= 0 & \text{on } (0,T)\times\partial\Omega, \\ (U,\partial_t U)|_{t=0} &= (U^0,U^1) & \text{in } \Omega, \end{cases} \tag{4.1.2}$$

with here

$$D = \begin{pmatrix} d_1 Id_{n_1} & 0 \\ 0 & d_2 Id_{n_2} \end{pmatrix}_{n\times n}, A = \begin{pmatrix} 0 & A_1 \\ 0 & A_2 \end{pmatrix}_{n\times n}, \text{ and } \hat{b} = \begin{pmatrix} 0 \\ b \end{pmatrix}_{n\times 1}, \tag{4.1.3}$$

where $n = n_1 + n_2$. Let us emphasize the following important and crucial properties of System (4.1.1): all coefficients are constant, the coupling is in a block-cascade structure (notably, the control f is only acting directly on U_2, which itself acts on U_1 through the matrix A_1), and we restrict to the case of a scalar control (*i.e.* $f \in L^2((0,T),\mathbb{R}^m)$ with $m = 1$). We will explain in conclusion the difficulties to treat more general cases.

4.1.2 Geometric assumptions

For our concerned domain Ω, we assume that Ω has no infinite order of tangential contact with the boundary. This assumption will be made more precise in Subsection 4.2.3. In fact, this assumption is sufficient to ensure the existence and uniqueness of the general bicharateristics passing through a given point in the phase space. Furthermore, for the control set ω, we assume the Geometric Control Condition (GCC).

Definition 4.1.1. *For $\omega \subset \Omega$ and $T > 0$, we shall say that the triple (ω, T, p) satisfies GCC if every generalized bicharacteristic of p meets ω in a time $t < T$, where p is the principal symbol of $\square$.*

We shall give a precise definition of the generalized bicharacteristics in Subsection 4.2.3. In the case of an internal control, GCC was firstly raised in [45] as a necessary condition for the controllability of the scalar wave equation from ω, and was proved to be sufficient in [8]. The case of a boundary control was studied in [10, 14].

4.1.3 Kalman conditions

In this part, we recall some Kalman rank conditions introduced in the literature of coupled parabolic systems and the link between them. First of all, we recall the usual Kalman rank condition for the controllability of linear autonomous ordinary differential equations (see *e.g.* [27]).

Definition 4.1.2 (Usual algebraic Kalman rank condition)**.** *Let m, n be two positive integers. Assume $X \in \mathcal{M}_n(\mathbb{R})$ and $Y \in \mathcal{M}_{n,m}(\mathbb{R})$. We introduce the Kalman matrix associated with X and Y given by $[X|Y] = [X^{n-1}Y|\cdots|XY|Y] \in \mathcal{M}_{n,nm}(\mathbb{R})$. We say that (X, Y) satisfies the Kalman rank condition if $[X|Y]$ has full rank.*

In order to generalize this usual algebraic Kalman rank condition, we introduce the Kalman operator (see [6]).

Definition 4.1.3 (Kalman operator)**.** *Assume that $X \in \mathcal{M}_n(\mathbb{R})$ and $Y \in \mathcal{M}_{n,m}(\mathbb{R})$. Moreover, let $D \in \mathcal{M}_n(\mathbb{R})$ be a diagonal matrix. Then, the Kalman operator associated with $(-D\Delta_D + X, Y)$ is the matrix operator $\mathscr{K} = [-D\Delta_D + X|Y] : \mathcal{D}(\mathscr{K}) \subset (L^2)^{nm} \to (L^2)^n)$, where the domain of the Kalman operator is given by $\mathcal{D}(\mathscr{K}) = \{u \in (L^2(\Omega))^{nm} : \mathscr{K}u \in (L^2(\Omega))^n\}$.*

Definition 4.1.4 (Operator Kalman rank condition)**.** *We say that the Kalman operator $\mathscr{K}$ satisfies the operator Kalman rank condition if $Ker(\mathscr{K}^*) = \{0\}$.*

The operator Kalman rank condition can be reformulated as follows.

Proposition 4.1.5. *[6, Proposition 2.2] The operator Kalman rank condition $Ker(\mathscr{K}^*) = \{0\}$ is equivalent to the following spectral Kalman rank condition:*

$$rank[(\lambda D + X)|Y] = n, \forall \lambda \in \sigma(-\Delta_D).$$

In particular, let $C > 0$ be a constant and $D = CId_n$. Then, the operator Kalman rank condition is equivalent to the usual algebraic Kalman rank conditionon (X, Y) given in Definition 4.1.2 *(see [6, Remark 1.2]).*

In the following proposition, we give an equivalent statement of the operator Kalman rank condition associated with System (4.1.1), which is very specific to our particular coupling structure and the fact that we have a single control.

Proposition 4.1.6. *We use the same notations $(D, A, \hat{b})$ as in* (4.1.3)*. We denote by $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ the Kalman operator associated with the System* (4.1.2)*. Then, $Ker(\mathcal{K}^*) = \{0\}$ is equivalent to satisfying all the following conditions:*

1. *$n_1 = 1$;*
2. *(A_2, b) satisfies the usual Kalman rank condition (See Definition* 4.1.2*);*
3. *Assume that $A_1 = \alpha = (\alpha_1, \cdots, \alpha_{n_2})$. Then, $\forall \lambda \in \sigma(-\Delta_D)$, α satisfies*

$$\alpha \left(\sum_{k=0}^{n_2-2} (d_1 - d_2)^k \lambda^k \sum_{j=k+1}^{n_2} a_j A_2^{j-1-k} + (d_1 - d_2)^{n_2-1} \lambda^{n_2-1} Id_{n_2} \right) b \neq 0, \tag{4.1.4}$$

where $(a_j)_{0 \le j \le n_2}$ are the coefficients of the characteristic polynomial $\chi(X)$ of the matrix A_2, i.e. $\chi(X) = X^{n_2} + \sum_{j=0}^{n_2-1} a_j X^j$, with the convention that $a_{n_2} = 1$.

We shall give the proof in Appendix 4.6.

Since we consider the control problem in a domain Ω with boundary, it is natural for us to introduce the following Hilbert spaces $H^s_\Omega(\Delta_D)$.

Definition 4.1.7. *We denote by $(\beta_j^2)_{j\in\mathbb{N}^*}$ the non-decreasing sequence of (positive) eigenvalues of $-\Delta_D$, repeated with multiplicity, and $(e_j)_{j\in\mathbb{N}^*}$ an orthonormal basis of $L^2(\Omega)$ made of eigenfunctions associated with $(\beta_j^2)_{j\in\mathbb{N}^*}$:*

$$-\Delta e_j = \beta_j^2 e_j,\ e_j(x) = 0, x\in\partial\Omega, \qquad ||e_j||_{L^2(\Omega)} = 1.$$

For any $s\in\mathbb{R}$, we denote by $H^s_\Omega(\Delta_D)$ the Hilbert space defined by

$$H^s_\Omega(\Delta_D) = \{u = \sum_{j\in\mathbb{N}^*} a_j e_j; \sum_{j\in\mathbb{N}^*} \beta_j^{2s}|a_j|^2 < \infty\}.$$

For convenience, we also denote

$$\mathscr{L}_s^k = (H^s_\Omega(\Delta_D))^k \text{ for any } s\in\mathbb{R}, \text{ and } k\in\mathbb{N}. \tag{4.1.5}$$

First, we give a necessary condition for the controllability of System (4.1.1).

Proposition 4.1.8. *We denote by $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ the Kalman operator associated with the System* (4.1.2)*. If $\mathcal{K}$ does not satisfy the operator Kalman rank condition, then System* (4.1.1) *is not null-controllable, in the following sense: there exists a quadruple*

$$(U_1^0, U_2^0, U_1^1, U_2^1) \in \bigcap_{s=1}^{+\infty} \left(\mathscr{L}_s^n \times \mathscr{L}_{s-1}^n\right)$$

such that for any control $f\in L^2(\omega)$, we necessarily have

$$(U(T,\cdot), \partial_t U(T,\cdot)) \neq (0,0).$$

We shall give the proof later in the Subsection 4.2.1.

From now on, we always assume that $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ satisfies the operator Kalman rank condition, so that we notably have $n_1 = 1$. Before we give a precise definition of the exact controllability property of System (4.1.1), we first investigate a simpler system. For a fixed $1\le s\le n_2$, we consider the following system

$$\begin{cases}
\Box_1 u_1^1 + \sum_{j=1}^s \alpha_s u_j^2 & = 0 & \text{in } (0,T)\times\Omega,\\
\Box_2 u_1^2 + u_2^2 & = 0 & \text{in } (0,T)\times\Omega,\\
\vdots & & \\
\Box_2 u_{n_2-1}^2 + u_{n_2}^2 & = 0 & \text{in } (0,T)\times\Omega,\\
\Box_2 u_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j} u_j^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega,\\
u_1^1 & = 0 & \text{on } (0,T)\times\partial\Omega,\\
u_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega, 1\le j\le n_2,\\
(u_1^1, u_1^2, \cdots, u_{n_2}^2)|_{t=0} & = (u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) & \text{in } \Omega,\\
(\partial_t u_1^1, \partial_t u_1^2, \cdots, \partial_t u_{n_2}^2)|_{t=0} & = (u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) & \text{in } \Omega.
\end{cases} \tag{4.1.6}$$

Here we have, $A_1 = (\alpha_1, \cdots, \alpha_s, 0, \cdots, 0)$ and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

The control is $f \in L^2((0,T) \times \omega)$. For this simpler system (4.1.6), taking zero initial conditions (that belong to any linear subspace and hence to any potential state space) together with a forcing term f in the space $L^2((0,T) \times \omega)$, which kind of target spaces will the solutions of System (4.1.6) arrive in? That is the first question we need to answer in order to be able to obtain an exact controllability result in an appropriate state space. Under this particular structure of coupling, we introduce appropriate compatibility conditions for System (4.1.6). For $r = 0, 1$, and $(u, v_1, \cdots, v_{n_2}) \in H_\Omega^{n_2-s+2+r}(\Delta_D) \times H_\Omega^{n_2-1+r}(\Delta_D) \times \cdots \times H_\Omega^r(\Delta_D)$, let us define a special function U^r_{comp} by

$$\begin{aligned} U^r_{comp} = \Big(& (-d_1\Delta)^{n_2-s+1} u \\ & + \sum_{k=0}^{n_2-s} \sum_{j=1}^{s} \sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l} (-d_1\Delta)^k (-d_2\Delta)^{n_2-s-k-l} v_{j+l} \\ & + \sum_{j=1}^{s} \sum_{k=0}^{n_2-2s+j} \sum_{l=0}^{n_2-s-k} \frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}} \binom{n_2-s-k}{l} (-d_2\Delta)^{n_2-s-k-l} v_{j+k+l} \Big). \end{aligned} \tag{4.1.7}$$

Using this special function U^r_{comp}, let us denote by $\mathcal{H}^s_r$ the following space:

$$\begin{aligned} \mathcal{H}^s_r = \{ & (u, v_1, \cdots, v_{n_2}) \in H_\Omega^{n_2-s+2+r}(\Delta_D) \times H_\Omega^{n_2-1+r}(\Delta_D) \times \cdots \times H_\Omega^r(\Delta_D) \\ & \text{s.t. } U^r_{comp} \in H^r_\Omega(\Delta_D) \}. \end{aligned} \tag{4.1.8}$$

Definition 4.1.9 (State space)**.** *The state space for System* (4.4.1) *is defined by*

$$\mathcal{H}^s_1 \times \mathcal{H}^s_0.$$

The two conditions

$$\begin{aligned} & U^1_{comp}(u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) \in H^1_\Omega(\Delta_D), \\ & U^0_{comp}(u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) \in H^0_\Omega(\Delta_D) \end{aligned}$$

are called the compatibility conditions for the controllability of System (4.4.1)*.*

Remark 4.1.10. *If $s = n_2$, the compatibility conditions reduce to*

$$-d_1 \Delta u_1^{1,0} \in H^1_\Omega(\Delta_D),$$
$$-d_1 \Delta u_1^{1,1} \in H^0_\Omega(\Delta_D),$$

which is an empty condition since we already know that $(u_0^1, u_1^1) \in H^3_\Omega(\Delta_D) \times H^2_\Omega(\Delta_D)$.

Remark 4.1.11. *As we will see later on, the solutions of System (4.1.6) will stay in $\mathcal{H}_1^s \times \mathcal{H}_0^s$ if the initial states are in this space. Because of the linearity and the time reversibility of the system, exact controllability is equivalent to null controllability or reachability from 0 for System (4.1.6). Since the equilibrium 0 is of course in the spaces $\mathcal{H}_1^s \times \mathcal{H}_0^s$, this is the appropriate state space.*

Remark 4.1.12. *Since we consider a system with a cascade coupling structure, it is natural that there is a gain of regularity for the uncontrolled states u_j^2 $(2 \leq j \leq n_2)$ (this phenomena has already been observed notably in [20, Theorem 1.4]). We shall explain the gain of two derivatives of regularity for the state u_1^1 in Subsection 4.2.2. We could call it "additional regularity".*

Now, we give the definition of the exact controllability of System (4.1.1).

Definition 4.1.13. *We say that System (4.1.1) is exactly controllable in time $T > 0$ if there exists $1 \leqslant s \leqslant n_2$ and $\mathcal{T} \in GL_n(\mathbb{R})$ such that for any initial data $(U_0, U_1) \in \mathcal{T}^{-1}(\mathcal{H}_1^s) \times \mathcal{T}^{-1}(\mathcal{H}_0^s)$ and any target $(\tilde{U}_0, \tilde{U}_1) \in \mathcal{T}^{-1}(\mathcal{H}_1^s) \times \mathcal{T}^{-1}(\mathcal{H}_0^s)$, there exists a control function $f \in L^2((0,T) \times \omega)$ such that the solution U of (4.1.1) satisfies $(U, \partial_t U)|_{t=0} = (U_0, U_1)$ and $(U, \partial_t U)|_{t=T} = (\tilde{U}_0, \tilde{U}_1)$, and $\mathcal{T}(U)$ is a solution of the associated System (4.1.6) with an appropriate control $\tilde{f}$.*

Remark 4.1.14. *By the definition above, in order to prove the controllability of System (4.1.1), we first look for an invertible transform to change the system into the simpler but equivalent System (4.1.6). Then, we prove the result for the simpler System (4.1.6) to conclude the exact controllability of the general System (4.1.1).*

Remark 4.1.15.

We shall see later that the transform $\mathcal{T}$ is just

$$\begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix},$$

where $P \in GL_{n_2}(\mathbb{R})$ is the transform associated with the Brunovský normal form defined in Theorem 4.3.1. Here we can give an example of the transform $\mathcal{T}$ under *a simple setting. If we consider a particular case of System* (4.1.1) *given by*

$$\begin{cases} \Box_1 u_1^1 - 2u_1^2 + u_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_1^2 + \frac{3}{2}u_1^2 - \frac{1}{2}u_2^2 & = 2f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ \Box_2 u_2^2 + \frac{9}{2}u_1^2 - \frac{3}{2}u_2^2 & = 4f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ u_1^1 & = 0 & \text{on } (0,T)\times\partial\Omega \\ u_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega,\ j=1,2, \\ (u_1^1, u_1^2, u_2^2)|_{t=0} & = (u_1^{1,0}, u_1^{2,0}, u_2^{2,0}) & \text{in } \Omega, \\ (\partial_t u_1^1, \partial_t u_1^2, \partial_t u_2^2)|_{t=0} & = (u_1^{1,1}, u_1^{2,1}, u_2^{2,1}) & \text{in } \Omega, \end{cases}$$

we have that

$$A = \begin{pmatrix} 0 & A_1 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{9}{2} & -\frac{3}{2} \end{pmatrix}, \text{ and } \hat{b} = \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}.$$

According to the Brunovský normal form, we obtain $\mathcal{T}_2 = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ *such that*

$$\mathcal{T}_2(A_2 b, b) = \mathcal{T}_2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the transform is given by $\mathcal{T} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{T}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$. *And moreover, this transform* $\mathcal{T}$ *satisfies*

$$\mathcal{T}\hat{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \mathcal{T}A\mathcal{T}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

There is a large literature on the controllability and observability of the wave equations. This paper is mainly devoted to multi-speed coupled wave systems. We list some of the existing results and references:

- For a single wave equation posed on a smooth bounded domain of $\mathbb{R}^d$ and with an internal control, one can use microlocal analysis to prove the observability inequality as done by Bardos, Lebeau and Rauch in [8]. We have two approaches to define the microlocal defect measures. We can introduce the microlocal defect measures based on the article by Gérard and Leichtnam [24] for Helmoltz equation and Burq [13] for the wave equation, using the extension by 0 across the boundary. On the other hand, we can also use the Melrose cotangent compressed bundle to construct the measure, based on the article by Lebeau [31] and Burq-Lebeau [15] in the setting of systems.

- Although we now have a better picture on the controllability of a single wave equation, the controllability of systems of wave equations is still not totally understood. To our knowledge, most of the references concern the case of systems with the same principal symbol $\Box$ on each equation of the system, which will be discussed in the present paragraph. Notably, Alabau-Boussouira and Léautaud [5] studied the indirect controllability of two coupled wave equations, in which their controllability result was established using a multi-level energy method introduced in [2], and also used in [3, 4]. Liard and Lissy [37] studied the observability and controllability for coupled wave systems with constant coefficients under Kalman type rank conditions. In the case of space-varying coefficients, Cui, Laurent, and Wang [19] studied the observability of wave equations coupled by space-varying first or zero order terms, on a compact manifold. Their results are extended to the case of manifold with boundaries in [18].
- Concerning the multi-speed case, Dehman, Le Roussau, and Léautaud considered two coupled wave equations on a compact manifold in [20]. Lissy and Zuazua [40] proved some general weak observability estimates for wave systems with constant or time-dependant coupling terms. Niu [44] investigated the case of the simultaneous controllability of wave systems, with different speeds and coupling terms involving only the controls, under various conditions on the speeds. Notably, in the case of constant speeds, a necessary and sufficient condition involving a Kalman rank condition was obtained, in the same spirit as in the present article.
- Concerning the boundary controllability of the coupled wave systems, we would like to refer to the works by Tatsien Li and Bopeng Rao, especially their work on the synchronisation of waves. In [32] and [33], Li and Rao for the first time studied the synchronization for systems described by PDEs. Taking a coupled system of wave equations with Dirichlet boundary controls as an example, they proposed the concept of exact boundary synchronization by boundary controls. After that, they and their collaborators successively got quite a lot of results (for instance, see [34, 36]). In particular, in [35], the authors obtain necessary conditions, presented as a criteria of Kalman's type, to the approximate null controllability, the approximate synchronization, respectively, for a coupled system of wave equations with Dirichlet boundary controls, which also show the link between the controllability of coupled wave systems and some appropriate Kalman conditions.

4.1.4 Main result

Our main result is the following one.

Theorem 4.1.16. *Given $T > 0$, suppose that:*

(i) (ω, T, p_{d_i}) *satisfies GCC, where* p_{d_i} *is the principal symbol of* $\Box_i, i = 1, 2$.

(ii) Ω *has no infinite order of tangential contact with the boundary.*

(iii) The Kalman operator $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ *associated with System* (4.1.1) *satisfies the operator Kalman rank condition, i.e.* $Ker(\mathcal{K}^*) = \{0\}$.

Then System (4.1.1) *is exactly controllable in the sense of Definition* 4.1.13.

Remark 4.1.17.

- *We will explain the concept of order of contact in the next section.*

- *Assume that conditions (i) and (ii) are verified. Then, condition (iii) is also necessary to have exact controllability in the sense of Definition* 4.1.13. *Indeed, if (iii) is not verified, Proposition* 4.1.8 *provides a smooth initial condition (that is notably in the state space introduced in Definition* 4.1.13*) that is not null-controllable.*

- *In fact, our proof also provides a controllability result for systems of wave equations with a single speed, of the form*

$$\begin{cases} \Box_2 U_2 + A_2 U_2 &= bf\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ U_2 &= 0 & \text{on } (0,T)\times\partial\Omega, \\ U_2|_{t=0} &= (U_1^0, U_2^0) & \text{in } \Omega, \\ \partial_t U_2|_{t=0} &= (U_1^1, U_2^1) & \text{in } \Omega. \end{cases} \tag{4.1.9}$$

  If (A_2, b) *does not verify the usual Kalman rank condition given in Definition 4.1.2, then this system is not exactly controllable in the same sense as in Proposition 4.1.8, with the same proof. If* (A_2, b) *verifies the usual Kalman rank condition, the state state space is*

$$P^{-1}(\tilde{\mathcal{H}}_1^s) \times P^{-1}(\tilde{\mathcal{H}}_0^s),$$

  where P is the transform associated with the Brunovský normal form defined in Theorem 4.3.1 *and* $\tilde{\mathcal{H}}_r^s$ $(r = 1, 2)$ *is given by*

$$\tilde{\mathcal{H}}_r^s = \{(v_1, \cdots, v_{n_2}) \in H_\Omega^{n_2-1+r}(\Delta_D) \times \cdots \times H_\Omega^r(\Delta_D)\}.$$

  Then, System (4.1.9) *is exactly controllable under this Kalman rank condition. This is a very particular case of the more general result proved in [18], where space-varying coefficients, multi-dimensional controls and also one-order coupling terms are considered.*

4.1.5 Outline of the chapter

The outline of this chapter is the following.

Section 4.2 is devoted to introducing some preliminaries. In Subsection 4.2.1, we present the necessity of the operator Kalman rank condition by giving the proof of Proposition 4.1.8. Then Subsection 4.2.2 is devoted to the "additional regularity" property for coupled wave equations. Subsection 4.2.3 includes the description of the boundary points, and give the precise definition of general bicharacteristics and the order of tangential contact with the boundary. Subsection 4.2.4 introduces the microlocal defect measures, which is the basic tool for our proof.

In Section 4.3, we focus on the special case $n_2 = 2$ to show the whole procedure of the proof of the controllability of the coupled wave system. Subsection 4.3.1 is devoted to reformulating the system with the help of the Brunovský normal form. Then in Subsection 4.3.2 we introduce the simpler system with one of the parameters being 0. We demonstrate the proof under this simple setting. In the following Subsection 4.3.3, we present the result of the general systems in a way very similar to the simpler case.

In Section 4.4, we plan to deal with any number of equations. Subsection 4.4.1 provides the corresponding simpler system in analogue with the Subsection 4.3.2 and gives the clear meaning of the compatibility conditions under the general setting. Then, with the help of the compatibility conditions, we are able to present the proof of the controllability result of Theorem 4.4.8. In the Subsection 4.4.2, we give the reformulation procedure of the general system.

In the concluding Section 4.5, we give some open problems related to our work, and explanations on the difficulties to solve them.

4.2 Preliminaries

We divide this section into four parts. The first part is devoted to proving the necessity of the operator Kalman rank condition. Then, we consider the regularities of the solutions of two coupled wave equations with different speeds. The third part aims to introduce the geometric preliminaries including the conceptions of general bicharacteristics and order of contact. The final part mainly contains the definition and some properties of the microlocal defect measures.

4.2.1 On the necessity of the operator Kalman rank condition

In this section, we are going to give the proof of Proposition 4.1.8. At first, we introduce the following proposition for the ordinary differential systems of second order.

Proposition 4.2.1. *If (A, b) does not satisfy the usual algebraic Kalman rank condition (see Definition* 4.1.2*), for any nonzero initial data $(y^0, y^1) \neq (0,0)$, the ordinary differential system*

$$\begin{cases} \frac{d^2y}{dt^2} & = A^*y \qquad \text{in } (0,T), \\ (y, \frac{dy}{dt})|_{t=0} & = (y^0, y^1), \end{cases} \tag{4.2.1}$$

has a nonzero solution satisfying $b^*y(t) = 0$ for every $t \in (0,T)$.

Proof. Define $z = \begin{pmatrix} y \\ \frac{dy}{dt} \end{pmatrix}$. Then, we are able to rewrite System (4.2.1) into a first-order system:

$$\begin{cases} \frac{dz}{dt} & = \tilde{A}^*z \qquad \text{in } (0,T) \\ z|_{t=0} = & {}^t(y^0, y^1), \end{cases} \tag{4.2.2}$$

where $\tilde{A} = \begin{pmatrix} 0 & A \\ Id_n & 0 \end{pmatrix}_{2n\times 2n}$. Let $\tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}_{2n\times 1}$. Easy computations give that

$$\tilde{A}^{2k} = \begin{pmatrix} A^k & 0 \\ 0 & A^k \end{pmatrix} \text{ and } \tilde{A}^{2k+1} = \begin{pmatrix} 0 & A^{k+1} \\ A^k & 0 \end{pmatrix} \text{ for } k = 0, 1, \cdots.$$

Therefore, we obtain

$$[\tilde{A}|\tilde{b}] = (\tilde{A}^{2n-1}\tilde{b}|\cdots|\tilde{A}\tilde{b}|\tilde{b}) = \begin{pmatrix} 0 & A^{n-1}b & \cdots & 0 & b \\ A^{n-1}b & 0 & \cdots & b & 0 \end{pmatrix}.$$

As a consequence, we know that $rank[\tilde{A}|\tilde{b}] = 2rank[A|b]$. Since (A, b) does not satisfy the usual algebraic Kalman rank condition, *i.e.*, $rank[A|b] < n$, we deduce that $rank[\tilde{A}|\tilde{b}] < 2n$, which implies that $(\tilde{A}, \tilde{b})$ does not satisfy the usual algebraic Kalman rank condition. By duality, this means that (4.2.2) is not observable through $\tilde{b}$.

Thus, there exists a nonzero solution $\zeta(t) = \begin{pmatrix} \zeta_1(t) \\ \zeta_2(t) \end{pmatrix} \in \mathbb{R}^{2n}$ to the associated adjoint system $\frac{dz}{dt} = \tilde{A}^*z$ satisfying that $\tilde{b}^*\zeta(t) = 0$ for every $t \in (0,T)$. Then, setting $y(t) = \zeta_1(t)$, we derive a nonzero solution $y(t)$ of System (4.2.1) satisfying that $b^*y(t) = b^*\zeta_1(t) = \tilde{b}^*\zeta(t) = 0$ for every $t \in (0,T)$. □

Now, we go back to the proof of Proposition 4.1.8.

Proof of Proposition 4.1.8. According to Proposition 4.1.5, since $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ does not satisfy the operator Kalman rank condition, there exists $\lambda_0 \in \sigma(-\Delta_D)$ such that $rank[(\lambda_0 D - A)|\hat{b}] < n$. As a consequence of Proposition 4.2.1, there exists a nonzero solution $\chi_{\lambda_0}(t) \in \mathbb{R}^n$ to the following ordinary differential system:

$$\begin{cases} \frac{d^2\chi}{dt^2} & = (\lambda_0 D - A^*)\chi & \text{in } (0,T), \\ (\chi, \frac{d\chi}{dt})|_{t=0} & = (\chi^0, \chi^1) \neq (0,0), \end{cases}$$

satisfying $\hat{b}^*\chi_{\lambda_0}(t) = 0$ for every $t \in (0,T)$. Then, let $\Phi(t,x) = \chi_{\lambda_0}(t)\varphi_{\lambda_0}(x)$, where φ_{λ_0} is an eigenfunction of $-\Delta_D$ associated with λ_0. Therefore, Φ satisfies the following system:

$$\begin{cases} (\partial_t^2 - D\Delta + A^*)\Phi & = 0 & \text{in } \Omega, \\ \hat{b}^*\Phi & = 0 & \text{for every } t \in (0,T), \\ \Phi|_{\partial\Omega} & = 0, \\ (\Phi, \partial_t\Phi)|_{t=0} & = (\chi^0\varphi_{\lambda_0}, \chi^1\varphi_{\lambda_0}) & \text{in } \Omega. \end{cases} \tag{4.2.3}$$

Suppose that there exists $f \in L^2((0,T)\times\omega)$ such that the corresponding solution U to (4.1.2) with initial state (U_0, U_1) satisfies

$$(U, \partial_t U)|_{t=T} = (0,0). \tag{4.2.4}$$

Then, by (4.1.2), we have

$$((\partial_t^2 - D\Delta_D + A)U, \Phi)_{L^2((0,T)\times\Omega)} = (\hat{b}f\mathbb{1}_\omega, \Phi)_{L^2((0,T)\times\Omega)}.$$

Integrating by parts on the left-hand side and using (4.2.3) together with (4.2.4) leads to

$$(U^0, \chi^1\varphi_{\lambda_0})_{L^2(\Omega)} - (U^1, -\chi^0\varphi_{\lambda_0})_{L^2(\Omega)} = (\hat{b}f\mathbb{1}_\omega, \Phi)_{L^2((0,T)\times\Omega)}.$$

Since $\widehat{b}^*\Phi = 0$ for every $t \in (0,T)$, we obtain that

$$(U^0, \chi^1\varphi_{\lambda_0})_{L^2(\Omega)} - (U^1, \chi^0\varphi_{\lambda_0})_{L^2(\Omega)} = 0.$$

Choosing $(U_0, U_1) = (\chi^1\varphi_{\lambda_0}, -\chi^0\varphi_{\lambda_0})$ leads to $(|\chi^1|^2 + |\chi^0|^2)\,||\varphi_{\lambda_0}||^2_{L^2(\Omega)} = 0$, which is a contradiction with $(\chi^0, \chi^1) \neq 0$. $\square$

4.2.2 On the regularity of coupled wave equations

Before investigating more complicated situations, let us concentrate on the regularity properties of the following simple system:

$$\begin{cases} \Box_1 u_1 + u_2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_2 & = f & \text{in } (0,T)\times\Omega, \\ u_1 = 0, u_2 & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=0} & = (u_1^0, u_1^1, u_2^0, u_2^1) & \text{in } \Omega. \end{cases} \tag{4.2.5}$$

Our next result gives a property of regularity for the solution of System (4.2.5). Such kind of extra regularity result was also observed in [20, Theorem 1.4], in which the authors stated the corresponding result in the case of a compact manifold without boundary. Here we will present a different (and more elementary) proof.

Lemma 4.2.2. *Assume that the initial conditions satisfy*

$$(u_1^0, u_1^1, u_2^0, u_2^1) \in H_\Omega^{\sigma+3}(\Delta_D) \times H_\Omega^{\sigma+2}(\Delta_D) \times H_\Omega^{\sigma+1}(\Delta_D) \times H_\Omega^{\sigma}(\Delta_D). \tag{4.2.6}$$

Then, there exists a unique solution to System (4.2.5) *satisfying*

$$\begin{aligned} u_1 &\in C^1([0,T], H_\Omega^{\sigma+2}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+3}(\Delta_D)), \\ u_2 &\in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D)). \end{aligned} \tag{4.2.7}$$

Proof. Since u_2 satisfies a wave equation with a source term $f \in L^1((0,T), H_\Omega^{\sigma}(\Delta_D))$, it is classical that there exists a unique solution

$$u_2 \in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D))$$

to the second line of System (4.2.5). Now, let us consider the first equation

$$\Box_1 u_1 = -u_2 \tag{4.2.8}$$

as a wave equation with a source term $u_2 \in L^1((0,T), H_\Omega^{\sigma+1}(\Delta_D))$. Thus, we know that there exists a unique solution $u_1 \in C^1([0,T], H_\Omega^{\sigma+1}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+2}(\Delta_D))$. Now, we need to state an extra regularity property for u_1. Applying the d'Alembert operator $\Box_2$ on both sides of (4.2.8), we obtain that

$$\Box_2 \Box_1 u_1 = -\Box_2 u_2.$$

Since $\Box_2 u_2 = f$, we know that $\Box_1(\Box_2 u_1) = -f$. We decompose $\Box_2 u_1$ into two parts $\Box_2 u_1 = \Box_1 u_1 + (d_1 - d_2)\Delta_D u_1$. Hence, we obtain that

$$\Box_2 u_1 = -u_2 + (d_1 - d_2)\Delta_D u_1. \tag{4.2.9}$$

Now, by using (4.2.6), we remark that the initial condition for $\Box_2 u_1$ verifies:

$$\begin{aligned} \Box_2 u_1|_{t=0} &= -u_2|_{t=0} + (d_1 - d_2)\Delta_D u_1|_{t=0} \\ &= -u_2^0 + (d_1 - d_2)\Delta_D u_1^0 \in H_\Omega^{\sigma+1}(\Delta_D), \end{aligned}$$

$$\begin{aligned} \partial_t(\Box_2 u_1)|_{t=0} &= -\partial_t u_2|_{t=0} + (d_1 - d_2)\Delta_D \partial_t u_1|_{t=0} \\ &= -u_2^1 + (d_1 - d_2)\Delta_D u_1^1 \in H_\Omega^{\sigma}(\Delta_D). \end{aligned}$$

So, we know that $\Box_2 u_1 \in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D))$. In addition, we also know that $-\Box_1 u_1 = u_2 \in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D))$. Hence, we obtain that

$$\Delta_D u_1 = \frac{1}{d_1 - d_2}(\Box_2 - \Box_1) u_1 \in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D)).$$

We conclude that $u_1 \in C^1([0,T], H_\Omega^{\sigma+2}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+3}(\Delta_D))$. $\Box$

4.2.3 Generalized bicharacteristics

***This part has many repeated contents as we have already presented in Section 2.3 of Chapter 1*.**

As usual, for a variable y, we denote $D_y = i\partial_y$. Let $B = \{y \in \mathbb{R}^d : |y| < 1\}$ be the unit euclidean ball in $\mathbb{R}^d$. In a tubular neighbourhood of the boundary, we can identify $M = \mathbb{R} \times \Omega$ locally as $X = (0,1) \times B$ and $\partial M = \mathbb{R} \times \partial\Omega$ locally as $\{0\} \times B$. Now, we consider $R = R(x, y, D_y)$ which is a second order scalar, self-adjoint, classical, tangential and smooth pseudo-differential operator, defined in a neighbourhood of $[0,1) \times B$ with a real principal symbol $r(x, y, \eta)$, such that

$$\frac{\partial r}{\partial \eta} \neq 0 \text{ for } (x,y) \in [0,1) \times B \text{ and } \eta \neq 0. \tag{4.2.10}$$

Let $Q_0(x, y, D_y)$, $Q_1(x, y, D_y)$ be smooth classical tangential pseudo-differential operators defined in a neighbourhood of $[0,1) \times B$, of order 0 and 1, and principal symbols $q_0(x, y, \eta)$, $q_1(x, y, \eta)$, respectively. Denote $P = (\partial_x^2 + R)Id + Q_0\partial_x + Q_1$. The principal symbol of P is

$$p = -\xi^2 + r(x, y, \eta). \tag{4.2.11}$$

We use the usual notations TM and T^*M to denote the tangent bundle and cotangent bundle corresponding to M, with the canonical projection π

$$\pi : TM(\text{ or } T^*M) \to M.$$

Denote $r_0(y, \eta) = r(0, y, \eta)$. Then, we can decompose $T^*\partial M$ into the disjoint union $\mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$, where

$$\mathcal{E} = \{r_0 < 0\}, \quad \mathcal{G} = \{r_0 = 0\}, \quad \mathcal{H} = \{r_0 > 0\}. \tag{4.2.12}$$

The sets $\mathcal{E}$, $\mathcal{G}$, $\mathcal{H}$ are called elliptic, glancing, and hyperbolic set, respectively. Define

$$\mathrm{Char}(P) = \{(x, y, \xi, \eta) \in T^*\mathbb{R}^{d+1}|_{\overline{M}} : \xi^2 = r(x, y, \eta)\} \tag{4.2.13}$$

to be the characteristic manifold of P. For more details, one can refer to [15] and [44]. Notice that in [13], one can see another characterization for these sets $\mathcal{E}$, $\mathcal{G}$, and $\mathcal{H}$.

To describe the different phenomena when a bicharacteristic approaches the boundary, we need a more accurate decomposition of the glancing set $\mathcal{G}$. Let

$r_1 = \partial_x r|_{x=0}$. Then, we can define the decomposition $\mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j$, with

$$\begin{aligned}
\mathcal{G}^2 &= \{(y,\eta) : r_0(y,\eta) = 0, r_1(y,\eta) \neq 0\}, \\
\mathcal{G}^3 &= \{(y,\eta) : r_0(y,\eta) = 0, r_1(y,\eta) = 0, H_{r_0}(r_1) \neq 0\}, \\
&\vdots \\
\mathcal{G}^{k+3} &= \{(y,\eta) : r_0(y,\eta) = 0, H_{r_0}^j(r_1) = 0, \forall j \leq k, H_{r_0}^{k+1}(r_1) \neq 0\}, \\
&\vdots \\
\mathcal{G}^{\infty} &= \{(y,\eta) : r_0(y,\eta) = 0, H_{r_0}^j(r_1) = 0, \forall j\}.
\end{aligned}$$

Here $H_{r_0}^j$ is just the Hamiltonian vector field H_{r_0} associated to r_0 composed j times. Moreover, for $\mathcal{G}^2$, we can define $\mathcal{G}^{2,\pm} = \{(y,\eta) : r_0(y,\eta) = 0, \pm r_1(y,\eta) > 0\}$. Thus $\mathcal{G}^2 = \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}$. For $\rho \in \mathcal{G}^{2,+}$, we say that ρ is a gliding point and for $\rho \in \mathcal{G}^{2,-}$, we say that ρ is a diffractive point. For $\rho \in \mathcal{G}^j$, $j \geq 2$, we say that a bicharacteristic of p tangentially contacts the boundary $\{x = 0\} \times B$ with order j at the point ρ.

We have the definition of the generalized bicharacteristics (See [26, Section 24.3] for more details):

Definition 4.2.3. *A generalized bicharacteristic of p is a map:*

$$s \in I \backslash D \mapsto \gamma(s) \in T^*M \cup \mathcal{G},$$

where I is an interval on $\mathbb{R}$ and D is a discrete subset I, such that $p \circ \gamma = 0$ and the following properties hold:

1. *$\gamma(s)$ is differentiable and $\frac{\mathrm{d}\gamma}{\mathrm{d}s} = H_p(\gamma(s))$ if $\gamma(s) \in T^*\overline{M} \backslash T^*\partial M$ or $\gamma(s) \in \mathcal{G}^{2,+}$.*

2. *Every $s \in D$ is isolated ,i.e., there exists $\epsilon > 0$ such that $\gamma(s) \in T^*\overline{M} \backslash T^*\partial M$ if $0 < |s - t| < \epsilon$, and the limits $\gamma(s^{\pm})$ are different points in the same fiber of $T^*\partial M$.*

3. *$\gamma(s)$ is differentiable and $\frac{\mathrm{d}\gamma}{\mathrm{d}s} = H_{-r_0}(\gamma(s))$ if $\gamma(s) \in \mathcal{G} \backslash \mathcal{G}^{2,+}$.*

Remark 4.2.4. *We denote the Melrose cotangent compressed bundle by ${}^bT^*\overline{M}$ and the associated canonical map by $j : T^*\overline{M} \mapsto {}^bT^*\overline{M}$. j is defined by*

$$j(x, y, \xi, \eta) = (x, y, x\xi, \eta). \tag{4.2.14}$$

Under this map j, one can see $\gamma(s)$ as a continuous flow on the compressed cotangent bundle ${}^bT^*\overline{M}$. This is the so-called Melrose-Sjöstrand flow (see [15] for more details).

From now on we always assume that there is no infinite tangential contact between the bicharacteristic of p and the boundary. This is in the meaning of the following definition:

Definition 4.2.5. *We say that there is no infinite contact between the bicharacteristics of p and the boundary if there exists $N \in \mathbb{N}$ such that the gliding set $\mathcal{G}$ satisfies*

$$\mathcal{G} = \bigcup_{j=2}^{N} \mathcal{G}^j.$$

It is well-known that under this hypothesis, there exists a unique generalized bicharacteristic passing through any point. This means that the Melrose-Sjöstrand flow is globally well-defined. One can refer to [42] and [43] for the proof.

4.2.4 Microlocal defect measure

In this section, we will give two approaches to construct the microlocal defect measures. The first one is based on the article by Gérard and Leichtnam [24] for Helmoltz equation and Burq [13] for wave equations. The other one follows the idea in the article [31] by Lebeau and we rely on the article [15] by Burq and Lebeau for the setting of wave systems. In the first approach, we can compare two different measures, especially the supports of two different measures. In the later proof, it is crucial to distinguish the measures with different speeds based on this idea. On the other hand, we use the second approach to describe the way the polarization of one measure is turning.

Let $(u^k)_{k\in\mathbb{N}}$ be a bounded sequence in $\left(L^2_{loc}(\mathbb{R}; L^2(\Omega))\right)^n$, converging weakly to 0 and such that

$$\begin{cases} Pu^k & = o(1)_{H^{-1}}, \\ u^k|_{x=0} & = 0. \end{cases}$$

Let $\underline{u}_k$ be the extension by 0 across $\{x = 0\}$. Then the sequence $\underline{u}_k$ is bounded in $\left(L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^d))\right)^n$. Let $\underline{\mathcal{A}}$ be the space of $n \times n$ matrices of classical polyhomogeneous pseudo-differential operators of order 0 with compact support in $\mathbb{R} \times \mathbb{R}^d$ (i.e, $A = \varphi A \varphi$ for some $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$). Let us denote by $\underline{\mathcal{M}}^+$ the set of nonnegative Radon measures on $T^*(\mathbb{R} \times \mathbb{R}^d)$. Following [13, Section 1], we have the existence of the microlocal defect measure as follows:

Proposition 4.2.6 (Existence of the microlocal defect measure-1)**.** *There exists a subsequence of $(\underline{u}^k)$ (still denoted by $(\underline{u}^k)$) and $\underline{\mu} \in \underline{\mathcal{M}}^+$ such that*

$$\forall A \in \underline{\mathcal{A}}, \quad \lim_{k\to\infty} (A\underline{u}^k, \underline{u}^k)_{L^2(\mathbb{R}\times\Omega)} = \langle \underline{\mu}, \sigma(A) \rangle, \tag{4.2.15}$$

where $\sigma(A)$ is the principal symbol of the operator A (which is a matrix of smooth functions, homogeneous of order 0 in the variable ξ).

From [13, Théorème 15], we have the following proposition.

Proposition 4.2.7. *For the microlocal defect measure $\underline{\mu}$ defined above, we have the following properties.*

- *The measure $\underline{\mu}$ is supported $Char(P) \cap (\mathbb{R} \times \overline{\Omega})$, where $Char(P)$ is defined in (4.2.13).*
- *The measure $\underline{\mu}$ does not charge the hyperbolic points in ∂M :*

$$\underline{\mu}(\mathcal{H}) = 0.$$

- *In particular, if $n = 1$, the scalar measure $\underline{\mu}$ is invariant along the generalized bicharacteristic flow.*

On the other hand, let $\mathcal{A}$ be the space of $n \times n$ matrices of pseudo-differential operators of order 0, in the form of $A = A_i + A_t$, with A_i a classical pseudo-differential operator with compact support in M($i.e$, $A_i = \varphi A_i \varphi$ for some $\varphi \in C_0^\infty(M)$) and A_t a classical tangential pseudo-differential operator in $\overline{M}$($i.e.$, $A_t = \varphi A_t \varphi$ for some $\varphi \in C^\infty(\overline{M})$). Then denote

$$Z = j(\mathrm{Char}(P)), \quad \hat{Z} = Z \cup j(T^*\overline{M}|_{x=0}),$$

where j is defined in (4.2.14) and

$$S\hat{Z} = (\hat{Z} \backslash \overline{M})/\mathbb{R}_+^*, \quad SZ = (Z \backslash \overline{M})/\mathbb{R}_+^*.$$

$S\hat{Z}$ and SZ are the quotient spherical spaces of $\hat{Z}$ and Z and they are locally compact metric spaces. Here, we identify the zero section $\overline{M} \times \{0\} \subset^b T^*\overline{M}$ with $\overline{M}$ itself.

For $A \in \mathcal{A}$, with principal symbol $a = \sigma(A)$, define

$$\kappa(a)(\rho) = a(j^{-1}(\rho)), \forall \rho \in {}^bT^*\overline{M}.$$

Now, we have that $\mathcal{K} = \{\kappa(a) : a = \sigma(A), A \in \mathcal{A}\} \subset C^0(S\hat{Z}; End(\mathbb{C}^n))$. Define $\mathcal{M}^+$ to be the space of all positive Borel measures on $S\hat{Z}$. By duality, we know that $\mathcal{M}^+$ is the dual space of $C_0^0(S\hat{Z}; End(\mathbb{C}^n))$, which verifies the property:

$$\langle \mu, a \rangle \geq 0, \forall a \in C^0(S\hat{Z}; End^+(\mathbb{C}^n)), \forall \mu \in \mathcal{M}^+,$$

where $End^+(\mathbb{C}^n)$ denotes the space of $n \times n$ positive hermitian matrices. Following the article [15] by Burq and Lebeau, we obtain the existence of the microlocal defect measure and some properties as follows:

Proposition 4.2.8 (Existence of the microlocal defect measure-2)**.** *There exists a subsequence of (u^k) (still noted by (u^k)) and $\mu \in \mathcal{M}^+$ such that*

$$\forall A \in \mathcal{A}, \quad \lim_{k\to\infty} (Au^k, u^k)_{L^2(\mathbb{R}\times\Omega)} = \langle \mu, \kappa(\sigma(A)) \rangle. \tag{4.2.16}$$

Lemma 4.2.9. *The microlocal defect measure μ defined in Proposition 4.2.8 satisfies that $\mu \mathbb{1}_{\mathcal{H}\cup\mathcal{E}} = 0$, where $\mathcal{H}$ is the set of hyperbolic points and $\mathcal{E}$ is the set of elliptic points as defined in Subsection 4.2.3.*

In the following, suppose that there is no infinite contact between the bicharacteristic of p and the boundary. This hypothesis implies the existence and uniqueness of the generalized bicharacteristic passing through any point, which ensures that the Melrose-Sjöstrand flow is globally well-defined. By a suitable change of parameter along this flow, we obtain a flow on SZ. Consider S a hypersurface tranverse to the flow. Then locally, $SZ = \mathbb{R}_s \times S$, where s is the well-chosen parameter along the flow. We have the following propagation lemma for the microlocal defect measure.

Lemma 4.2.10. *Assume that the microlocal defect measure μ is defined in Proposition 4.2.8. Then μ is supported in SZ and there exists a function*

$$(s, z) \in \mathbb{R}_s \times S \mapsto M(s, z) \in \mathbb{C}^n$$

$\mu-$almost everywhere continuous such that the pullback of the measure μ by M (i.e., the measure $\mathcal{P}^*\mu = M^*\mu M$ defined for $a \in C^0(SZ)$) by

$$\langle M^*\mu M, a \rangle = \langle \mu, MaM^* \rangle$$

satisfies

$$\frac{d}{ds}\mathcal{P}^*\mu = 0.$$

We say that the measure μ is invariant along the flow associated with M. Furthermore, the function M is continuous, and along any generalized bicharacteristic, the matrix M is solution to a differential equation whose coefficients can be explicitly computed in terms of the geometry and the different terms in the operator P.

For the differential equation that M satisfies, one can refer to [15, Section 3.2] for more details.

Remark 4.2.11. *Roughly speaking, in the result above, the Frobenius norm of M describes the damping of the measure μ, whereas the rotation component of M (i.e. the orthogonal part of the polar decomposition) describes the way the polarization of the measure (asymptotic polarization of the sequence (u^k)) is turning.*

Remark 4.2.12. *Notice that in [13, Section 3], the author considered the case of solutions to the wave equation at the energy level (bounded in H^1_{loc}), and hence was considering second order operators. However, it is easy to change the energy level into L^2, one can see [44, Remark 4.4] for more details.*

Remark 4.2.13. *From Proposition 4.2.7, we know that supp($\underline{\mu}$) $\subset$ Char(P). Notice that in the interior of M, the two definitions coincide, i.e., for any pseudo-differential operator A of order 0 with principal symbokl $\sigma(A)$ satisfying supp($\sigma(A)$) $\subset$ Char$(P)|_M$, we have $\langle \underline{\mu}, \sigma(A)\rangle = \langle \mu, \kappa(\sigma(A))$, simply by their definitions. At the boundary, since both measures $\underline{\mu}$ and μ do not not charge the hyperbolic points in ∂M, we know that $\underline{\mu}|_{S\hat{Z}} = \mu$ holds μ almost surely and $\underline{\mu}$ almost surely. Under this sense, we can identify the two measures.*

4.3 Proof of the sufficient part of Theorem 4.1.16 in the case $n_2 = 2$

In this section, we shall present the sufficient part of the proof of Theorem 4.1.16 in the case $n_2 = 2$ (and of course $n_1 = 1$). We divide the proof into three steps. Firstly, we give a reformulation of System (4.3.1). Then we study a simpler problem and obtain a compatibility condition for it. At last, we present the proof for the general case.

4.3.1 Reformulation of the system in symmetric spaces

In the case $n_2 = 2$, we write System (4.1.1) as follows:

$$
.\begin{cases}
\partial_t^2 u_1^1 - d_1 \Delta u_1^1 + \alpha_1 u_1^2 + \alpha_2 u_2^2 & = 0 \qquad \text{in } (0,T)\times\Omega,\\
\partial_t^2 u_1^2 - d_2 \Delta u_1^2 + a_{11} u_1^2 + a_{12} u_2^2 & = b_1 f \mathbb{1}_\omega \quad \text{in } (0,T)\times\Omega,\\
\partial_t^2 u_2^2 - d_2 \Delta u_2^2 + a_{21} u_1^2 + a_{22} u_2^2 & = b_2 f \mathbb{1}_\omega \quad \text{in } (0,T)\times\Omega,\\
u_1^1 = 0, u_j^2 & = 0 \qquad \text{on } (0,T)\times\partial\Omega,\ j = 1,2,
\end{cases}
\tag{4.3.1}
$$

with initial conditions

$$
(u_1^1(0,x), u_1^2(0,x), u_2^2(0,x), \partial_t u_1^1(0,x), \partial_t u_1^2(0,x), \partial_t u_2^2(0,x))
$$

belonging to a space that will be detailed later on.

Before we reformulate the system, we introduce the Brunovský normal form.

Theorem 4.3.1 (Brunovský Normal Form)**.** *Assume that A is a square matrix of size $n \times n$, B is a matrix of size $n \times 1$ and (A, B) satisfies the Kalman rank* *condition. Then, there exists an invertible matrix P such that $A = P^{-1}JP$ and $B = P^{-1}e_n$, where*

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{pmatrix}, \text{ and } e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{4.3.2}$$

and the coefficients $(a_j)_{1\le j\le n}$ are defined by the characteristic polynomial of A, i.e. $\chi_A(X) = X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n$.

One can find for instance the proof in [50, Théorème 2.2.7] for this theorem. Now, we set $\tilde{A}$, $\tilde{B}$, and α by

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tilde{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ and } \alpha = (\alpha_1, \alpha_2).$$

Then, we obtain $A = \begin{pmatrix} 0 & \alpha \\ 0 & \tilde{A} \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \tilde{B} \end{pmatrix}$. As a consequence of (4.1.6), we know that $(\tilde{A}, \tilde{B})$ satisfies the Kalman rank condition. Hence, by the Brunovský normal form, there exists an invertible matrix $\tilde{P}$ such that

$$\tilde{A} = \tilde{P}\begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix}\tilde{P}^{-1}, \tilde{B} = \tilde{P}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and } \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) = \alpha\tilde{P}^{-1}.$$

Furthermore, according to the third statement of Proposition 4.1.6, we know that

$$\tilde{\alpha}_2(d_1 - d_2)\lambda + \tilde{\alpha}_1 \neq 0, \forall \lambda \in \sigma(-\Delta_D). \tag{4.3.3}$$

Using the change of unknowns

$$\begin{pmatrix} \tilde{u}_1^1 \\ \tilde{u}_1^2 \\ \tilde{u}_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{P} \end{pmatrix}\begin{pmatrix} u_1^1 \\ u_1^2 \\ u_2^2 \end{pmatrix}, \tag{4.3.4}$$

we obtain a simplified system

$$\begin{cases} \Box_1 \tilde{u}_1^1 + \tilde{\alpha}_1\tilde{u}_1^2 + \tilde{\alpha}_2\tilde{u}_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 \tilde{u}_1^2 + \tilde{u}_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 \tilde{u}_2^2 - a_1\tilde{u}_1^2 - a_2\tilde{u}_2^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ \tilde{u}_1^1 = 0, \tilde{u}_1^2 = 0, \tilde{u}_2^2 & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\tilde{u}_1^1(0,x), \tilde{u}_1^2(0,x), \tilde{u}_2^2(0,x))|_{t=0} & = (\tilde{u}_1^{1,0}, \tilde{u}_1^{2,0}, \tilde{u}_2^{2,0}) & \text{in } \Omega, \\ (\partial_t\tilde{u}_1^1(0,x), \partial_t\tilde{u}_1^2(0,x), \partial_t\tilde{u}_2^2(0,x))|_{t=0} & = (\tilde{u}_1^{1,1}, \tilde{u}_1^{2,1}, \tilde{u}_2^{2,1}) & \text{in } \Omega. \end{cases} \tag{4.3.5}$$

Therefore, the exact controllability of System (4.3.1) is equivalent to the exact controllability of System (4.3.5). Classically, given the initial conditions

$$(\tilde{u}_1^{2,0}, \tilde{u}_2^{2,0}, \tilde{u}_1^{2,1}, \tilde{u}_2^{2,1}) \in H^2_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^0_\Omega(\Delta_D),$$

the solutions $\tilde{u}_1^2$ and $\tilde{u}_2^2$ satisfy

$$\begin{aligned}\tilde{u}_1^2 &\in C^0([0,T], H^2_\Omega(\Delta_D)) \cap C^1([0,T], H^1_\Omega(\Delta_D)),\\ \tilde{u}_2^2 &\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)),\end{aligned}$$

As for the regularity of the solution $\tilde{u}_1^1$, it depends on the coupling term $\tilde{\alpha}_1\tilde{u}_1^2 + \tilde{\alpha}_2\tilde{u}_2^2$. Thus, it is natural to discuss in two different cases, *i.e.* $\tilde{\alpha}_2 \neq 0$ and $\tilde{\alpha}_2 = 0$.

4.3.2 The case $\tilde{\alpha}_2 = 0$

In what follows, we will present into details the proof of Theorem 4.1.16 firstly in the case $n_2 = 2$ (and $n_1 = 1$ by Proposition 4.1.6), and $A_1 = (\alpha_1, 0)$. Here, for the sake of simplicity we remove the $\tilde{}$ in our notations and we investigate the system

$$\begin{cases} \Box_1 u_1^1 + \alpha_1 u_1^2 = 0 & \text{in } (0,T)\times\Omega,\\ \Box_2 u_1^2 + u_2^2 = 0 & \text{in } (0,T)\times\Omega,\\ \Box_2 u_2^2 - a_1 u_1^2 - a_2 u_2^2 = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega,\\ u_1^1 = 0, u_j^2 = 0 & \text{on } (0,T)\times\partial\Omega,\ j=1,2,\\ (u_1^1, u_1^2, u_2^2)|_{t=0} = (u_1^{1,0}, u_1^{2,0}, u_2^{2,0}) & \text{in } \Omega,\\ (\partial_t u_1^1, \partial_t u_1^2, \partial_t u_2^2)|_{t=0} = (u_1^{1,1}, u_1^{2,1}, u_2^{2,1}) & \text{in } \Omega. \end{cases} \tag{4.3.6}$$

For this system, we have the following well-posedness property.

Proposition 4.3.2. *Assume that the initial conditions satisfy*

$$\begin{aligned}(u_1^{2,0}, u_2^{2,0}, u_1^{2,1}, u_2^{2,1}) &\in H^2_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^0_\Omega(\Delta_D),\\ (u_1^{1,0}, u_1^{1,1}) &\in H^4_\Omega(\Delta_D) \times H^3_\Omega(\Delta_D).\end{aligned}$$

Additionally, assume that

$$(-\Delta_D)^2 u_1^{1,0} - \frac{\alpha_1}{d_1 - d_2}\Delta_D u_1^{2,0} \in H^1_\Omega(\Delta_D),\ \ (-\Delta_D)^2 u_1^{1,1} - \frac{\alpha_1}{d_1 - d_2}\Delta_D u_1^{2,1} \in H^0_\Omega(\Delta_D). \tag{4.3.7}$$

Then, the solutions u_1^1, u_1^2 and u_2^2 satisfy

$$\begin{aligned} u_1^1 &\in C^0([0,T], H^4_\Omega(\Delta_D)) \cap C^1([0,T], H^3_\Omega(\Delta_D)),\\ u_1^2 &\in C^0([0,T], H^2_\Omega(\Delta_D)) \cap C^1([0,T], H^1_\Omega(\Delta_D)),\\ u_2^2 &\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)),\\ (-\Delta_D)^2 u_1^1 - \frac{\alpha_1}{d_1 - d_2}\Delta_D u_1^2 &\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)).\end{aligned}$$

Proof of Proposition 4.3.2. Classically, given the initial conditions

$$(u_1^{2,0}, u_2^{2,0}, u_1^{2,1}, u_2^{2,1}) \in H^2_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^0_\Omega(\Delta_D),$$

the solutions u_1^2 and u_2^2 satisfy

$$\begin{aligned} u_1^2 &\in C^0([0,T], H^2_\Omega(\Delta_D)) \cap C^1([0,T], H^1_\Omega(\Delta_D)), \\ u_2^2 &\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)). \end{aligned} \tag{4.3.8}$$

According to Lemma 4.2.2, given the initial condition

$$u_1^{1,0}, u_1^{1,1} \in H^4_\Omega(\Delta_D) \times H^3_\Omega(\Delta_D),$$

the solution u_1^1 satisfies

$$u_1^1 \in C^0([0,T], H^4_\Omega(\Delta_D)) \cap C^1([0,T], H^3_\Omega(\Delta_D)). \tag{4.3.9}$$

Let us first do some reformulation for the system. Define the transform $\mathcal{S}_0$ by

$$\mathcal{S}_0 \begin{pmatrix} u_1^1 \\ u_1^2 \\ u_2^2 \end{pmatrix} = \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_2^2 \end{pmatrix}, \tag{4.3.10}$$

where

$$\begin{cases} v_1^1 = D_t^3 u_1^1, \\ v_1^2 = D_t u_1^2, \\ v_2^2 = u_2^2. \end{cases} \tag{4.3.11}$$

We need to invert the previous relations by expressing u_1^1, u_1^2, u_2^2 in terms of v_1^1, v_1^2, v_2^2. Firstly, for the term $u_2^2 = v_2^2$, there is nothing to do. Then, we look at the term u_1^2. We need to "invert" in some sense the operator D_t. We use the second equation of System (4.3.6). We apply D_t on the second equation of System (4.3.11), and we obtain

$$\begin{aligned} D_t v_1^2 &= D_t^2 u_1^2 \\ &= u_2^2 - d_2 \Delta u_1^2 \\ &= v_2^2 - d_2 \Delta u_1^2. \end{aligned}$$

Hence, we obtain that

$$u_1^2 = \frac{(-\Delta_D)^{-1}}{d_2}(D_t v_1^2 - v_2^2). \tag{4.3.12}$$

For the last term u_1^1, we apply D_t on the first equation of System (4.3.11), then we use the first equation of System (4.3.6), the second equation of System (4.3.6) and the last equation of System (4.3.11) to obtain

$$\begin{aligned}
D_t v_1^1 &= D_t^2(D_t^2 u_1^1) \\
&= \alpha_1 D_t^2 u_1^2 - d_1 \Delta D_t^2 u_1^1 \\
&= \alpha_1(u_2^2 - d_2\Delta u_1^2) - d_1\Delta_D(\alpha_1 u_1^2 - d_1\Delta u_1^1) \\
&= (-d_1\Delta)^2 u_1^1 - \alpha_1(d_1+d_2)\Delta u_1^2 + \alpha_1 v_2^2.
\end{aligned}$$

Therefore, from the above computations, (4.3.11), and (4.3.12), an inverse transform is the following:

$$\begin{cases}
u_1^1 = \frac{(-\Delta_D)^{-2}}{d_1^2}(D_t v_1^1 + \alpha_1 \frac{d_1+d_2}{d_2} D_t v_1^2 + \alpha_1 \frac{d_1}{d_2} v_2^2), \\
u_1^2 = \frac{(-\Delta_D)^{-1}}{d_2}(D_t v_1^2 - v_2^2), \\
u_2^2 = v_2^2.
\end{cases} \tag{4.3.13}$$

From the regularity results given in (4.3.8), (4.3.9) and the relations (4.3.13), we obtain that

$$\begin{aligned}
v_1^1 &\in C^0([0,T]; H_\Omega^1(\Delta_D)) \cap C^1([0,T]; H_\Omega^0(\Delta_D)), \\
v_j^2 &\in C^0([0,T]; H_\Omega^1(\Delta_D)) \cap C^1([0,T]; H_\Omega^0(\Delta_D)),\ j=1,2.
\end{aligned} \tag{4.3.14}$$

Moreover, from (4.3.6) and (4.3.13), (v_1^1, v_1^2, v_2^2) satisfies the following system:

$$\begin{cases}
\Box_1 v_1^1 + \alpha_1 D_t^2 v_1^2 & = 0 & \text{in } (0,T)\times\Omega, \\
\Box_2 v_1^2 + D_t v_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\
\Box_2 v_2^2 - \frac{a_1(-\Delta_D)^{-1}}{d_2}(D_t v_1^2 - v_2^2) - a_2 v_2^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\
v_1^1 = 0, v_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega,\ j=1,2,
\end{cases} \tag{4.3.15}$$

with appropriate initial conditions. Using the identity

$$-D_t^2 = \frac{1}{d_2 - d_1}\left(d_2\Box_1 - d_1\Box_2\right), \tag{4.3.16}$$

we obtain that

$$D_t^2 v_1^2 = -\frac{1}{d_2-d_1}(d_2\Box_1 - d_1\Box_2)v_1^2. \tag{4.3.17}$$

Using (4.3.17) in the first equation of (4.3.15), we also deduce that

$$\Box_1\left(v_1^1 - \frac{\alpha_1 d_2}{d_2-d_1}v_1^2\right) - \frac{\alpha_1 d_1}{d_2-d_1}D_t v_2^2 = 0. \tag{4.3.18}$$

Now, let us define

$$y = D_t v_1^1 - \frac{\alpha_1 d_2}{d_2-d_1}D_t v_1^2. \tag{4.3.19}$$

Then, by (4.3.19) and (4.3.18), we obtain that

$$\Box_1 y - \frac{\alpha_1 d_1}{d_2 - d_1} D_t^2 v_2^2 = 0. \tag{4.3.20}$$

We also remark that by using (4.3.16),

$$-D_t^2 v_2^2 = \frac{1}{d_2 - d_1}(d_2\Box_1 - d_1\Box_2)v_2^2. \tag{4.3.21}$$

Using the last equation of (4.3.15) together with (4.3.20) and (4.3.21), we deduce that

$$\Box_1 \left(y + \frac{\alpha_1 d_1 d_2}{(d_2 - d_1)^2} v_2^2 \right) = \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} f + \frac{\alpha_1 a_1 d_1^2 (-\Delta_D)^{-1}}{d_2 (d_2 - d_1)^2}(D_t v_1^2 - v_2^2) + \frac{\alpha_1 a_2 d_1^2}{(d_2 - d_1)^2} v_2^2. \tag{4.3.22}$$

Let us now express y with respect to the original variables u_1^1, u_1^2, u_2^2. From (4.3.19), (4.3.11) and the first equation of (4.3.6), we obtain that

$$\begin{aligned} y &= D_t v_1^1 - \frac{\alpha_1 d_2}{d_2 - d_1} D_t v_1^2 \\ &= D_t^4 u_1^1 - \frac{\alpha_1 d_2}{d_2 - d_1} D_t^2 u_1^2 \\ &= D_t^2 \left(D_t^2 u_1^1 - \frac{\alpha_1 d_2}{d_2 - d_1} u_1^2 \right) \\ &= D_t^2 \left(-d_1 \Delta u_1^1 + \alpha_1 u_1^2 - \frac{\alpha_1 d_2}{d_2 - d_1} u_1^2 \right) \\ &= D_t^2 \left(-d_1 \Delta u_1^1 - \frac{\alpha_1 d_1}{d_2 - d_1} u_1^2 \right). \end{aligned} \tag{4.3.23}$$

Combining with the second equation of (4.3.6), we obtain

$$y = (-d_1\Delta)^2 u_1^1 - \frac{\alpha_1 d_1^2}{d_1 - d_2} \Delta u_1^2 + \frac{\alpha_1 d_1}{d_1 - d_2} u_2^2.$$

Hence, we obtain

$$y + \frac{\alpha_1 d_2 d_1}{(d_1 - d_2)^2} u_2^2 = (-d_1\Delta)^2 u_1^1 - \frac{\alpha_1 d_1^2}{d_1 - d_2} \Delta u_1^2 + \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} u_2^2.$$

Now, we define

$$\tilde{y} = y + \frac{\alpha_1 d_2 d_1}{(d_1 - d_2)^2} u_2^2.$$

Then, $\tilde{y}$ satisfies

$$\Box_1 \tilde{y} = \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} f + \frac{\alpha_1 a_1 d_1^2 (-\Delta_D)^{-1}}{d_2(d_2-d_1)^2}(D_t v_1^2 - v_2^2) + \frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2} v_2^2. \tag{4.3.24}$$

The initial condition associated with $\tilde{y}$ is given by

$$\begin{aligned}
\tilde{y}|_{t=0} &= \left((-d_1\Delta)^2 u_1^1 - \frac{\alpha_1 d_1^2}{d_1-d_2}\Delta u_1^2 + \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} u_2^2\right)|_{t=0} \\
&= (-d_1\Delta)^2 u_1^{1,0} - \frac{\alpha_1 d_1^2}{d_1-d_2}\Delta u_1^{2,0} + \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} u_2^{2,0} \\
&= d_1^2\left((-\Delta)^2 u_1^{1,0} - \frac{\alpha_1}{d_1-d_2}\Delta u_1^{2,0}\right) + \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} u_2^{2,0} \\
\partial_t \tilde{y}|_{t=0} &= \left((-d_1\Delta)^2 \partial_t u_1^1 - \frac{\alpha_1 d_1^2}{d_1-d_2}\Delta \partial_t u_1^2 + \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} \partial_t u_2^2\right)|_{t=0} \\
&= (-d_1\Delta)^2 u_1^{1,1} - \frac{\alpha_1 d_1^2}{d_1-d_2}\Delta u_1^{2,1} + \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} u_2^{2,1} \\
&= d_1^2\left((-\Delta)^2 u_1^{1,1} - \frac{\alpha_1}{d_1-d_2}\Delta u_1^{2,1}\right) + \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} u_2^{2,1}.
\end{aligned}$$

Hence, from our Hypothesis (4.3.7) together with (4.3.8) and (4.3.9), we deduce that

$$\tilde{y}|_{t=0} \in H^1_\Omega(\Delta_D), \;\; \partial_t \tilde{y}|_{t=0} \in H^0_\Omega(\Delta_D). \tag{4.3.25}$$

By (4.3.24) and (4.3.14), $\tilde{y}$ satisfies a wave equation with a source term in the space $L^1((0,T), H^0_\Omega(\Delta_D))$ and initial condition in $H^1_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D)$ by (4.3.25). We deduce that

$$\tilde{y} \in C^0([0,T]; H^1_\Omega(\Delta_D)) \cap C^1([0,T]; H^0_\Omega(\Delta_D)).$$

Hence, from (4.3.24) and (4.3.23), we deduce that

$$(-\Delta)^2 u_1^1 - \frac{\alpha_1}{d_1-d_2}\Delta u_1^2 + \frac{\alpha_1}{(d_2-d_1)^2} u_2^2 \in C^0([0,T]; H^1_\Omega(\Delta_D)) \cap C^1([0,T]; H^0_\Omega(\Delta_D)).$$

Taking into account the last line of (4.3.8), this implies that

$$(-\Delta)^2 u_1^1 - \frac{\alpha_1}{d_1-d_2}\Delta u_1^2 \in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)).$$

■

Remark 4.3.3. *Let us define the transform $\mathcal{S}$ associated with the system* (4.3.6) *and* (4.3.26) *by*

$$\mathcal{S}\begin{pmatrix} u_1^1 \\ u_1^2 \\ u_2^2 \end{pmatrix} = \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_2^2 \end{pmatrix},$$

where

$$\mathcal{S} = \begin{pmatrix} (-d_1\Delta_D)^2 & -\frac{\alpha_1 d_1^2}{d_1-d_2}\Delta_D & \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} \\ 0 & D_t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and its "inverse"

$$\mathcal{S}^{-1} = \begin{pmatrix} \frac{(-\Delta_D)^{-2}}{d_1^2} & -\frac{\alpha_1(-\Delta_D)^{-2}}{d_2(d_1-d_2)}D_t & \frac{\alpha_1(d_1-2d_2)(-\Delta_D)^{-2}}{d_2(d_1-d_2)^2}) \\ 0 & \frac{(-\Delta_D)^{-1}}{d_2}D_t & -\frac{(-\Delta_D)^{-1}}{d_2} \\ 0 & 0 & 1 \end{pmatrix}.$$

The previous computations show that we have a bijection between the solutions of (4.3.6) *and* (4.3.26). *Notably, if* $U = \begin{pmatrix} u_1^1 \\ u_1^2 \\ u_2^2 \end{pmatrix}$ *and* $V = \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_2^2 \end{pmatrix}$, *then* $\mathcal{S} \circ \mathcal{S}^{-1} V = V$ *and* $\mathcal{S}^{-1} \circ \mathcal{S} U = U$.

Notably, (4.3.6) *can be rewritten as*

$$(\partial_t^2 - D\Delta + A)(\mathcal{S}^{-1} \circ \mathcal{S}U) = \hat{b}f.$$

Therefore, since $\mathcal{S}(U) = V$ *we are able to rewrite the system* (4.3.26) *as follows:*

$$(\partial_t^2 - \mathcal{S}D\mathcal{S}^{-1}\Delta + \mathcal{S}A\mathcal{S}^{-1})V = \mathcal{S}\hat{b}f,$$

where

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_2 \end{pmatrix}, \ A = \begin{pmatrix} 0 & \alpha_1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_1 & -a_2 \end{pmatrix}, \ \mathcal{S}\hat{b}f = \begin{pmatrix} \frac{\alpha_s d_1^2}{(d_1-d_2)^2} f \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix}.$$

Moreover, we could notice that both $\mathcal{S}$ and $\mathcal{S}^{-1}$ only involve D_t and $(-\Delta_D)^k, k \in \mathbb{Z}$. This abstract point of view will be useful in the proof of the general case given in Section 4.4.

Now, we consider the exact controllability of System (4.3.6) in the space $\mathcal{H}_1^1 \times \mathcal{H}_0^0$, according to Proposition 4.3.2.

We have the following result:

Theorem 4.3.4. *Given $T > 0$, suppose that:*

1. *(ω, T, p_{d_i}) satisfies GCC, $i = 1, 2$.*
2. *Ω has no infinite order of tangential contact with the boundary.*

Then System (4.3.6) is exactly controllable in $\mathcal{H}_1^1 \times \mathcal{H}_0^0$.

Recall that here the state space $\mathcal{H}_1^1 \times \mathcal{H}_0^0$ is given by

$$\begin{aligned}
\mathcal{H}_1^1 &= \{(u, v_1, v_2) \in H_\Omega^4(\Delta_D) \times H_\Omega^2(\Delta_D) \times H_\Omega^1(\Delta_D),\\
&\quad (-d_1\Delta)^2 u - \frac{\alpha_1 d_1^2}{d_1 - d_2}\Delta v_1 \in H_\Omega^1(\Delta_D)\},\\
\mathcal{H}_0^0 &= \{(u, v_1, v_2) \in H_\Omega^3(\Delta_D) \times H_\Omega^1(\Delta_D) \times H_\Omega^0(\Delta_D),\\
&\quad (-d_1\Delta)^2 u - \frac{\alpha_1 d_1^2}{d_1 - d_2}\Delta v_1 \in H_\Omega^0(\Delta_D)\}.
\end{aligned}$$

Proof of Theorem 4.3.4.

By the computations of Proposition 4.3.2, proving Theorem 4.3.4 is equivalent to proving the exact controllability of the following system:

$$\begin{cases}
\Box_1 v_1^1 - \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2}(D_t v_1^2 - v_2^2) - \frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2} v_2^2 & = \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} f \mathbb{1}_\omega \quad \text{in } (0,T)\times\Omega,\\
\Box_2 v_1^2 + D_t v_2^2 & = 0 \quad \text{in } (0,T)\times\Omega,\\
\Box_2 v_2^2 - \frac{a_1(-\Delta_D)^{-1}}{d_2}(D_t v_1^2 - v_2^2) - a_2 v_2^2 & = f\mathbb{1}_\omega \quad \text{in } (0,T)\times\Omega,\\
v_1^1 & = 0 \quad \text{on } (0,T)\times\partial\Omega,\\
v_1^2 = v_2^2 & = 0 \quad \text{on } (0,T)\times\partial\Omega,
\end{cases} \tag{4.3.26}$$

with initial conditions

$$\begin{aligned}
(v_1^1, v_1^2, v_2^2)|_{t=0} &\in (H_0^1(\Omega))^3 = \mathscr{L}_1^3,\\
(\partial_t v_1^1, \partial_t v_1^2, \partial_t v_2^2)|_{t=0} &\in (L^2(\Omega))^3 = \mathscr{L}_0^3,
\end{aligned}$$

in the state space $\mathscr{L}_1^3 \times \mathscr{L}_0^3$. Recall that we defined $\mathscr{L}_s^k = (H_\Omega^s(\Delta_D))^k$ in (4.1.5). According to the Hilbert Uniqueness Method of J.-L. Lions [38], the exact controllability of System (4.3.26) is equivalent to proving the following observability inequality: there exists $C > 0$ such that for any solution of the adjoint system:

$$\begin{cases}
\Box_1 w_1^1 & = 0 \quad \text{in } (0,T)\times\Omega,\\
\Box_2 w_1^2 - \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2} D_t w_1^1 - \frac{a_1(-\Delta_D)^{-1}}{d_2} D_t w_2^2 & = 0 \quad \text{in } (0,T)\times\Omega,\\
\Box_2 w_2^2 + D_t w_1^2 - a_2 w_2^2 + \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2} w_1^1 & \\
-\frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2} w_1^1 + \frac{a_1(-\Delta_D)^{-1}}{d_2} w_2^2 & = 0 \quad \text{in } (0,T)\times\Omega,\\
w_1^1 & = 0 \quad \text{on } (0,T)\times\partial\Omega,\\
w_1^2 = w_2^2 & = 0 \quad \text{on } (0,T)\times\partial\Omega,
\end{cases} \tag{4.3.27}$$

with initial conditions

$$(w_1^1, w_1^2, w_2^2)|_{t=0} \in \mathscr{L}_0^3, \tag{4.3.28}$$

$$(\partial_t w_1^1, \partial_t w_1^2, \partial_t w_2^2)|_{t=0} \in \mathscr{L}_{-1}^3, \tag{4.3.29}$$

we have the following observability inequality:

$$C \int_0^T \int_\omega \left| \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} w_1^1 + w_2^2 \right|^2 dxdt \geq ||W(0)||^2_{\mathscr{L}_0^3 \times \mathscr{L}_{-1}^3}, \tag{4.3.30}$$

where $W = (w_1^1, w_1^2, w_2^2)$.

Remark 4.3.5. *As we showed in Remark* 4.3.3*, we are able to rewrite the system* (4.3.27) *as follows:*

$$(\partial_t^2 - (\mathcal{S}')^{-1} D \mathcal{S}' \Delta + (\mathcal{S}')^{-1} A^* \mathcal{S}') W = 0.$$

However, we should pay attention to this $\mathcal{S}'$, which is defined as the invertible transform between two adjoint systems. $\mathcal{S}'$ could be seen as the "adjoint" operator of $\mathcal{S}$. To be more specific, we write the original adjoint system as follows:

$$\begin{cases} \Box_1 z_1^1 & = 0 \quad \textit{in } (0,T) \times \Omega, \\ \Box_2 z_1^2 + \alpha_1 z_1^1 - a_1 z_2^2 & = 0 \quad \textit{in } (0,T) \times \Omega, \\ \Box_2 z_2^2 + z_1^2 - a_2 z_2^2 & = 0 \quad \textit{in } (0,T) \times \Omega, \\ z_1^1 = 0, z_j^2 & = 0 \quad \textit{on } (0,T) \times \partial\Omega, \ j = 1,2. \end{cases} \tag{4.3.31}$$

The transform $\mathcal{S}'$ associated with the system (4.3.27) *and* (4.3.31) *is defined by*

$$\mathcal{S}' \begin{pmatrix} w_1^1 \\ w_1^2 \\ w_2^2 \end{pmatrix} = \begin{pmatrix} z_1^1 \\ z_1^2 \\ z_2^2 \end{pmatrix},$$

where

$$\mathcal{S}' = \begin{pmatrix} (-d_1 \Delta_D)^2 & 0 & 0 \\ -\frac{\alpha_1 d_1^2}{d_1 - d_2} \Delta_D + \frac{a_1 \alpha_1 d_1^2 (-\Delta_D)^{-1}}{d_2 (d_1 - d_2)^2} & D_t & \frac{a_1 (-\Delta_D)^{-1}}{d_2} \\ \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} & 0 & 1 \end{pmatrix}, \tag{4.3.32}$$

and its "inverse" by

$$(\mathcal{S}')^{-1} = \begin{pmatrix} (-d_1 \Delta_D)^{-2} & 0 & 0 \\ -\frac{\alpha_1 (-\Delta_D)^{-2}}{d_2 (d_1 - d_2)} D_t & (-d_2 \Delta_D)^{-1} D_t & 0 \\ -\frac{\alpha_1 (-\Delta_D)^{-2}}{(d_2 - d_1)^2} & 0 & 1 \end{pmatrix}.$$

Moreover, we could notice that both $\mathcal{S}'$ and $(\mathcal{S}')^{-1}$ only involve D_t and $(-\Delta_D)^k, k \in \mathbb{Z}$. As already written, this point of view will be useful in the proof of the general case given in Section 4.4.

We divide the proof of the observability inequality (4.3.30) into two steps.

Step 1: establish a relaxed observability inequality.

Firstly, we establish the following relaxed observability inequality for the adjoint System (4.3.27).

Proposition 4.3.6. *For solutions of System* (4.3.27)*, there exists a constant* $C > 0$ *such that for any solution of* (4.3.27) *with initial conditions verifying* (4.3.28)*, we have*

$$||W(0)||^2_{\mathscr{L}^3_0\times\mathscr{L}^3_{-1}} \leq C\left(\int_0^T\int_\omega\left|\frac{\alpha_1 d_1^2}{(d_2-d_1)^2}w_1^1+w_2^2\right|^2 dxdt + ||W(0)||^2_{\mathscr{L}^3_{-1}\times\mathscr{L}^3_{-2}}\right). \tag{4.3.33}$$

Proof of Proposition 4.3.6. We argue by contradiction. Suppose that the observability inequality (4.3.33) is not satisfied. Thus, there exists a sequence $(W^k)_{k\in\mathbb{N}}$ of solutions of System (4.3.27) such that

$$||W^k(0)||^2_{\mathscr{L}^3_0\times\mathscr{L}^3_{-1}} = 1, \tag{4.3.34}$$

$$\int_0^T\int_\omega\left|\frac{\alpha_1 d_1^2}{(d_2-d_1)^2}w_1^{1,k}+w_2^{2,k}\right|^2 dxdt \to 0 \text{ as } k\to\infty, \tag{4.3.35}$$

$$||W^k(0)||^2_{\mathscr{L}^3_{-1}\times\mathscr{L}^3_{-2}} \to 0 \text{ as } k\to\infty. \tag{4.3.36}$$

By the continuity of the solution with respect to the initial data of System (4.3.27), we know that the sequence $(W^k)_{k\in\mathbb{N}}$ is bounded in $(L^2((0,T)\times\Omega))^3$ and moreover, $W^k \rightharpoonup 0$ in $(L^2((0,T)\times\Omega))^3$. W^k satisfies the following system:

$$\begin{cases} \Box_1 w_1^{1,k} & = o(1)_{H^{-1}} \quad \text{in } (0,T)\times\Omega,\ k\to\infty \\ \Box_2 w_1^{2,k} & = o(1)_{H^{-1}} \quad \text{in } (0,T)\times\Omega,\ k\to\infty \\ \Box_2 w_2^{2,k}+D_t w_1^{2,k} & = o(1)_{H^{-1}} \quad \text{in } (0,T)\times\Omega,\ k\to\infty, \end{cases} \tag{4.3.37}$$

where the first equation is decoupled from the two last equations.

Remark 4.3.7. *We say* $f^k = o(1)_{H^{-1}}$ *if* $\lim_{k\to\infty}||f^k||_{H^{-1}((0,T)\times\Omega)} = 0$*. Let us explain briefly how to obtain* (4.3.37)*. We take the term* $\frac{(-\Delta_D)^{-1}}{d_2}D_t w_2^{2,k}$ *for instance. Other terms can be treated similarly. For* $\frac{(-\Delta_D)^{-1}}{d_2}D_t w_2^{2,k}$*, we know that* $\frac{(-\Delta_D)^{-1}}{d_2}D_t w_2^{2,k} \in L^2((0,T);H^2_\Omega)\cap H^{-1}((0,T);H^1_\Omega)$ *is a bounded sequence and converges weakly to* 0*. Since the injection from* $L^2((0,T);H^2_\Omega)\cap H^{-1}((0,T);H^1_\Omega)$ *to* $H^{-1}((0,T)\times\Omega)$ *is compact, we obtain that* $\frac{(-\Delta_D)^{-1}}{d_2}D_t w_2^{2,k} = o(1)_{H^{-1}}$.

Hence, we obtain two microlocal defect measures $\underline{\mu}_1$ and $\underline{\mu}_2$ associated with $(w_1^{1,k})_{k\in\mathbb{N}}$ and $(W^{2,k})_{k\in\mathbb{N}} = (w_1^{2,k}, w_2^{2,k})_{k\in\mathbb{N}}$ respectively. From the definition in

Proposition 4.2.6, we know that

$$\begin{aligned} \forall A \in \underline{\mathcal{A}}, \quad \langle \underline{\mu}_1, \sigma(A)\rangle &= \lim_{k\to\infty} (A\underline{w}_1^{1,k}, \underline{w}_1^{1,k})_{L^2}, \\ \langle \underline{\mu}_2(i,j), \sigma(A)\rangle &= \lim_{k\to\infty} (A\underline{w}_i^{2,k}, \underline{w}_j^{2,k})_{L^2}, 1 \le i,j \le 2. \end{aligned} \tag{4.3.38}$$

Here $\underline{\mu}_2 = (\underline{\mu}_2(i,j))_{1\le i,j\le 2}$ is the matrix measure associated with the sequence $(W^{2,k})_{k\in\mathbb{N}} = (w_1^{2,k}, w_2^{2,k})_{k\in\mathbb{N}}$ and $\underline{w}_i^{j,k}$ is the extension by 0 across the boundary of Ω ($1 \le i,j \le 2$). Moreover, since the two characteristic manifolds $\mathrm{Char}(p_{d_1})$ and $\mathrm{Char}(p_{d_2})$ are compact and disjoint, $\underline{\mu}_1$ and $\underline{\mu}_2$ are mutually singular in $(0,T)\times\Omega$, from the first point of Proposition 4.2.7. Therefore, we obtain the following property:

Lemma 4.3.8. *For $A \in \underline{\mathcal{A}}$ with compact support in $(0,T)\times\Omega$ and for $1 \le i \le 2$, we have*

$$\limsup_{k\to\infty} |(A\underline{w}_1^{1,k}, \underline{w}_i^{2,k})_{L^2(\mathbb{R}\times\Omega)}| = 0. \tag{4.3.39}$$

Proof. We follow the same strategy as for the proof of [44, Lemma 4.10]. Since $\mathrm{Char}(p_{d_1})$ and $\mathrm{Char}(p_{d_2})$ are disjoint, we choose a cut-off function $\beta \in C^\infty(T^*\mathbb{R}\times\mathbb{R}^d)$ homogeneous of degree 0 for $|(\tau,\xi)| \ge 1$, with compact support in $(0,T)\times\Omega$ such that

$$\beta|_{\mathrm{Char}(p_{d_1})} = 1, \beta|_{\mathrm{Char}(p_{d_2})} = 0, \text{ and } 0 \le \beta \le 1.$$

Since $A \in \underline{\mathcal{A}}$ with compact support in $(0,T)\times\Omega$, for some $\varphi \in C_0^\infty((0,T)\times\omega)$, we have that $A = \varphi A\varphi$. We introduce $\tilde{\varphi} \in C_0^\infty((0,T)\times\omega)$ such that $\tilde{\varphi}|_{\mathrm{supp}(\varphi)} = 1$ i.e, $\tilde{\varphi}\varphi = \varphi$. Now, let us consider $(A\underline{w}_1^{1,k}, \underline{w}_2^{2,k})_{L^2}$. First, we have that

$$\begin{aligned} (A\underline{w}_1^{1,k}, \underline{w}_2^{2,k})_{L^2} &= (\varphi A\varphi\underline{w}_1^{1,k}, \underline{w}_2^{2,k})_{L^2} \\ &= (\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2} \\ &= ((1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2} + (\mathrm{Op}(\beta)\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2}. \end{aligned}$$

For the first term $((1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2}$, by the Cauchy-Schwarz inequality, we obtain that

$$|((1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2}| \le ||(1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}||_{L^2}||\tilde{\varphi}\underline{w}_2^{2,k}||_{L^2}. \tag{4.3.40}$$

As we know that $\{\underline{w}_2^{2,k}\}$ is bounded in $L^2_{loc}(\mathbb{R}^+\times\mathbb{R}^d)$, there exists a constant C such that

$$||\tilde{\varphi}\underline{w}_2^{2,k}||_{L^2}^2 = (\tilde{\varphi}\underline{w}_2^{2,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2} \le C. \tag{4.3.41}$$

From the definition of the measure $\underline{\mu}_1$, we obtain

$$\begin{aligned} \lim_{k\to\infty} ||(1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}||_{L^2}^2 &= \lim_{k\to\infty} ((1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}, (1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k})_{L^2} \\ &= \langle \underline{\mu}_1, (1-\beta)^2\varphi^4|\sigma(A)|^2\rangle. \end{aligned} \tag{4.3.42}$$

From Proposition 4.2.7, we have that supp $(\underline{\mu}_1) \subset \mathrm{Char}(p_{d_1})$. In addition, by the choice of β, we know that $1-\beta \equiv 0$ on supp $(\underline{\mu}_1)$, which implies that $\langle \underline{\mu}_1, (1-\beta)^2\varphi^4|\sigma(A)|^2\rangle = 0$. Combining (4.3.40), (4.3.41) and (4.3.42), we obtain

$$\limsup_{k\to\infty} |((1-\mathrm{Op}(\beta))\varphi A\varphi \underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2}| = 0. \tag{4.3.43}$$

The other term is dealt with similarly. One can refer to [44, Lemma 4.10] for more details. □

Let us go back to the proof of Proposition 4.3.6. We know that

$$\int_0^T \int_\omega \left| \frac{d_1^2}{(d_2-d_1)^2} w_1^{1,k} + w_2^{2,k} \right|^2 dxdt \to 0 \text{ as } k \to \infty.$$

For $\chi \in C_0^\infty(\omega \times (0,T))$, by expending the above expression,

$$\begin{aligned} &2(\frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}, \chi w_2^{2,k})_{L^2(\mathbb{R}\times\Omega)} \\ &+(\frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}, \frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k})_{L^2(\mathbb{R}\times\Omega)} + (\chi w_2^{2,k}, \chi w_2^{2,k})_{L^2(\mathbb{R}\times\Omega)} \to 0, \text{ as } k \to \infty. \end{aligned}$$

By Lemma 4.3.8, we know that

$$\limsup_{k\to\infty} \left|(\frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}, \chi w_2^{2,k})_{L^2(\mathbb{R}\times\Omega)}\right| = 0.$$

As a consequence, since we know that

$$\left| \frac{d_1^2}{(d_2-d_1)^2} w_1^{1,k} + w_2^{2,k} \right|^2 \geqslant 0,$$

we deduce that

$$\begin{aligned} (\frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}, \frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k})_{L^2(\mathbb{R}\times\Omega)} \to 0, \\ (\chi w_2^{2,k}, \chi w_2^{2,k})_{L^2(\mathbb{R}\times\Omega)} \to 0, \text{ as } k \to \infty. \end{aligned}$$

Thus, using (4.3.38), we know that (here $\mu_2 = (\mu_2(i,j))_{1\leq i,j\leq 2}$ is a matrix measure)

$$\underline{\mu}_1|_{(0,T)\times\omega} = 0, \text{ and } \underline{\mu}_2(2,2)|_{(0,T)\times\omega} = 0.$$

For $\underline{\mu}_1$, since $\underline{\mu}_1$ is invariant along the general bicharacteristics of p_{d_1}, combining with GCC, we obtain as usual that $\underline{\mu}_1 \equiv 0$. For $\underline{\mu}_2$, we consider another definition of the microlocal defect measure. From the definition in Proposition 4.2.8, we know that there exists a measure μ_2 such that

$$\forall A \in \mathcal{A}, \quad \langle \mu_2, \kappa(\sigma(A)) \rangle = \lim_{k\to\infty} (AW^{2,k}, W^{2,k})_{L^2}. \tag{4.3.44}$$

Since $\underline{\mu}_2|_{Char(p_{d_2})} = \mu_2$ μ_2-almost surely by Remark 4.2.13, we obtain that $\mu_2(2,2)|_{(0,T)\times\omega} = 0$. In the following part, we aim to prove that $\mu_2 = 0$. The basic idea is to use Lemma 4.2.10. Here we recall this lemma under our setting of this adjoint system.

Lemma 4.3.9. *Assume that μ_2 is the corresponding microlocal defect measure defined by* (4.3.44) *for the sequence $(w_1^{2,k}, w_2^{2,k})_{k\in\mathbb{N}}$ which satisfies the following system (according to* (4.3.27)*):*

$$\begin{cases} \Box_2 w_1^{2,k} & = o(1)_{H^{-1}} \quad \textit{in } (0,T)\times\Omega,\ k\to\infty \\ \Box_2 w_2^{2,k} + D_t w_1^{2,k} & = o(1)_{H^{-1}} \quad \textit{in } (0,T)\times\Omega,\ k\to\infty. \end{cases} \tag{4.3.45}$$

If we denote the general bicharacteristic by $s \mapsto \gamma(s)$, then along $\gamma(s)$ there exists a continuous function $s \mapsto M(s)$ such that M satisfies the differential equation:

$$\frac{d}{ds}(M(s)) = iE(\tau)M(s), M(0) = Id,$$

and μ_2 is invariant along the flow associated with M, which means that

$$\frac{d}{ds}(M^*\mu_2 M) = 0.$$

Here we denote by $E(\tau)$ the matrix $\begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix}$.

Remark 4.3.10. *For the differential equation which M satisfies and the explicit form of the matrix E which we use here, one can refer to [15, Section 3.2] for more details.*

Remark 4.3.11. *In our setting, we can compute explicitly the form of the matrix*

$$M(s) = \begin{pmatrix} 1 & i\tau s \\ 0 & 1 \end{pmatrix}$$

and τ is a constant with respect to s along the generalized bicharacteristic by the explicit form of $Char(P)$ given in (4.2.13).

Now we use this Lemma 4.3.9 to prove that $\mu_2 = 0$. First, we would like to show that $supp(\mu_2) \cap \pi^{-1}((0,T)\times\omega) = \emptyset$. Let us fix some point $\rho_0 \in \pi^{-1}((0,T)\times\omega)$. Then, there exists a unique bicharacteristic $s \mapsto \gamma_0(s)$ such that $\gamma_0(0) = \rho_0$.

Moreover, there exists $\epsilon > 0$, which is sufficiently small, such that $\gamma_0((-2\epsilon, 2\epsilon)) \subset \pi^{-1}((0,T)\times\omega)$. Since μ_2 is invariant along the flow associated with M, we obtain $\mu_2(0) = M(\epsilon)^*\mu_2(\epsilon)M(\epsilon)$. Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By a straightforward computation using the special form of M, we have

$$M(\epsilon)e_2 = i\tau\epsilon M(\epsilon)e_1 + e_2.$$

Hence, we obtain

$$\begin{aligned} \mu_2(0)e_2 &= M(\epsilon)^*\mu_2(\epsilon)M(\epsilon)e_2 \\ &= M(\epsilon)^*\mu_2(\epsilon)(i\tau\epsilon M(\epsilon)e_1 + e_2) \\ &= i\tau\epsilon\mu_2(0)e_1 + M(\epsilon)^*\mu_2(\epsilon)e_2. \end{aligned} \tag{4.3.46}$$

We know that $\mu_2(2,2) \equiv 0$ on $(0,T)\times\omega$, which means that $w^k_{2,2} \to 0$ strongly in $L^2((0,T)\times\omega)$. Hence, by (4.3.38), we also have that $\mu_2(\epsilon)e_2 = 0$. Hence, we obtain $\mu_2(0)e_2 = -i\tau\epsilon\mu(0)e_1$. But by the choice of ρ_0, we know that $\mu_2(0)e_2$ also vanishes, which gives that $-i\tau\epsilon\mu_2(0)e_1 = 0$, *i.e.* $\mu_2(0)e_1 = 0$. Hence, $\mu_2(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since ρ_0 is arbitrary, we deduce that $\mathrm{supp}(\mu_2) \cap \pi^{-1}((0,T)\times\omega) = \emptyset$.

Now, let us go back to prove that $\mu_2 = 0$. For any point $\rho_1 \in \mathrm{supp}(\mu_2)$, there exists a unique bicharacteristic $s \mapsto \gamma_1(s)$ such that $\gamma_1(0) = \rho_1$. Using the GCC (see Definition 4.1.1), we know that there exists a time t_0 such that $\gamma_1(t_0) \in \pi^{-1}((0,T)\times\omega)$. Since μ_2 is invariant along the flow associated with M, we obtain

$$\mu_2(0) = M(t_0)^*\mu_2(t_0)M(t_0). \tag{4.3.47}$$

We already know that $\mathit{supp}(\mu_2) \cap \pi^{-1}((0,T)\times\omega) = \emptyset$, which means that $\mu_2(t_0) = 0$. By (4.3.47), we deduce that $\mu_2(0) = 0$. Due to the arbitrary choice of ρ_1, we obtain that $\mathit{supp}(\mu_2) = \emptyset$, *i.e.* $\mu_2 \equiv 0$, which leads to a contradiction with (4.3.34) (See [44, Section 4.2] for more details). We conclude that the relaxed observability inequality (4.3.33) holds for all the solutions of System (4.3.27). ∎

Step 2: analysis of the invisible solutions

With the relaxed observability inequality (4.3.33) in Proposition 4.3.6, we are now able to handle the low-frequencies and conclude the proof of the observability (4.3.30). The main point here is a unique continuation result for solutions of the elliptic problem associated with System (4.3.27). The idea of reducing the observability for the low frequencies to an elliptic unique continuation result and associated technology are due to [10]. First, let us write for the sake of simplicity the initial conditions as

$$\mathscr{W} = (w_1^{1,0}, w_1^{2,0}, w_2^{2,0}, w_1^{1,1}, w_1^{2,1}, w_2^{2,1})^t \ (\in \mathscr{L}_0^3 \times \mathscr{L}_{-1}^3), \tag{4.3.48}$$

and define for any $T > 0$ the set of invisible solutions (see [10]) from $(0,T) \times \omega$

$$\mathscr{N}_3(T) = \{\mathscr{W} \in \mathscr{L}_0^3 \times \mathscr{L}_{-1}^3 \text{ such that the associated solution of System (4.3.27)}$$
$$\text{satisfies } \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} w_1^1(x,t) + w_2^2(x,t) = 0, \forall (x,t) \in (0,T) \times \omega\}.$$

We have the following key lemma, which is proved at the end of this section.

Lemma 4.3.12. $\mathscr{N}_3(T) = \{0\}$.

Assume for the moment that Lemma 4.3.12 holds. As for the proof of the observability inequality (4.3.30), we proceed by contradiction. If the observability inequality (4.3.30) were false, we could find a sequence $(W^k)_{k\in\mathbb{N}}$ of solutions to System (4.3.27) which satisfy

$$||W^k(0)||^2_{\mathscr{L}_0^3 \times \mathscr{L}_{-1}^3} = 1, \tag{4.3.49}$$

$$\int_0^T \int_\omega \left| \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} w_1^{1,k} + w_2^{2,k} \right|^2 dxdt \to 0 \text{ as } k \to \infty. \tag{4.3.50}$$

By the well-posedness, we know that $(W^k)_{k\in\mathbb{N}}$ is bounded in $L^2((0,T)\times\Omega)$. Hence, there exists a subsequence (also denoted by W^k) weakly converging in $L^2((0,T) \times \Omega)$, towards $W \in L^2((0,T) \times \Omega)$, which is also a solution of System (4.3.27) (since what we consider is a linear system) and satisfies that $\frac{\alpha_1 d_1^2}{(d_2-d_1)^2} w_1^1 + w_2^2 = 0$ in $(0,T) \times \omega$. Thus, we know that $W(0) \in \mathscr{N}(T) = \{0\}$, which implies that $W(0) = 0$. Since the embedding $L^2 \times H_\Omega^{-1}(\Delta_D) \hookrightarrow H_\Omega^{-1}(\Delta_D) \times H_\Omega^{-2}(\Delta_D)$ is compact, we obtain that $||W^k(0)||^2_{\mathscr{L}_{-1}^3 \times \mathscr{L}_{-2}^3} \to ||W(0)||^2_{\mathscr{L}_{-1}^3 \times \mathscr{L}_{-2}^3}$. From the relaxed observability inequality (4.3.33), we know that

$$1 \leq C||W(0)||^2_{\mathscr{L}_{-1}^3 \times \mathscr{L}_{-2}^3},$$

which contradicts to the fact that $W(0) = 0$. Then we can conclude the observability inequality (4.3.30).

It only remains to prove Lemma 4.3.12.

Proof of Lemma 4.3.12. According to the relaxed observability inequality (4.3.33), for $\mathscr{W} \in \mathscr{N}(T)$, we obtain that

$$||W(0)||^2_{\mathscr{L}_0^3 \times \mathscr{L}_{-1}^3} \leq C||W(0)||^2_{\mathscr{L}_{-1}^3 \times \mathscr{L}_{-2}^3}. \tag{4.3.51}$$

We know that $\mathcal{N}(T)$ is a closed subspace of $\mathscr{L}_0^3 \times \mathscr{L}_{-1}^3$. By the compact embedding $L^2(\Omega) \times H^{-1}(\Omega) \hookrightarrow H^{-1}(\Omega) \times H^{-2}(\Omega)$, we know that $\mathcal{N}(T)$ has a finite dimension. Then, we define the operator $\mathscr{A}$ as

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -d_1\Delta_D & 0 & 0 & 0 & 0 & 0 \\ 0 & -d_2\Delta_D & 0 & -\frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2} & 0 & -\frac{a_1(-\Delta_D)^{-1}}{d_2} \\ \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2} - \frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2} & 0 & -d_2\Delta_D - a_2 + \frac{a_1(-\Delta_D)^{-1}}{d_2} & 0 & 1 & 0 \end{pmatrix}.$$

We know that the solution $(w_1^1, w_1^2, w_2^2, D_t w_1^1, D_t w_1^2, D_t w_2^2)^t$ can be written as

$$\begin{pmatrix} w_1^1 \\ w_1^2 \\ w_2^2 \\ D_t w_1^1 \\ D_t w_1^2 \\ D_t w_2^2 \end{pmatrix} = e^{-t\mathscr{A}}\mathscr{W},$$

where $\mathscr{W}$ is defined in (4.3.48). Let $\delta \in (0,T)$, we know that (4.3.51) is still true for $\mathscr{W} \in \mathscr{N}(T-\delta)$. Taking $\mathscr{W} \in \mathscr{N}(T)$, for $\epsilon \in]0,\delta[$, we have $e^{-\epsilon\mathscr{A}}\mathscr{W} \in \mathscr{N}(T-\delta)$. For α large enough, as $\epsilon \to 0^+$,

$$(\alpha+\mathscr{A})^{-1}\frac{1}{\epsilon}(Id - e^{-\epsilon\mathscr{A}})\mathscr{W} \to (\alpha+\mathscr{A})^{-1}\mathscr{A}\mathscr{W} \text{ as } \varepsilon \to 0^+ \text{Â in } \mathscr{L}_0^3 \times \mathscr{L}_{-1}^3. \quad (4.3.52)$$

Remind that

$$D(\mathscr{A}) = \{U \in \mathscr{L}_0^3 \times \mathscr{L}_{-1}^3 | \frac{d}{dt}(e^{-t\mathscr{A}})_{t=0^+} \text{ converges}\}. \quad (4.3.53)$$

Since $||(\alpha+\mathscr{A})^{-1}\cdot||_{\mathscr{L}_0^3\times\mathscr{L}_{-1}^3}$ is a norm, (4.3.52) means that $(Id - e^{-\epsilon\mathscr{A}})_{\varepsilon>0}$ is convergent for this norm. Since all norms are equivalent on the finite-dimensional linear subspace $\mathscr{N}(T)$, we notably deduce that $(Id - e^{-\epsilon\mathscr{A}})\mathscr{W}$ converges in $\mathscr{L}_0^3 \times \mathscr{L}_{-1}^3$, so that $\mathscr{W} \in D(\mathscr{A})$ by (4.3.53). We deduce that $N(T-\delta) \subset D(\mathscr{A})$. Since this equality is true for any $\delta \in (0,T)$, we deduce that $N(T) \subset D(\mathscr{A})$. Hence, for $\mathscr{W} \in N(T)$, we have

$$\frac{d}{dt}(e^{-t\mathscr{A}}(\mathscr{W}))_{t=0^+} = -\mathscr{A}\mathscr{W}.$$

Since $\mathcal{N}(T)$ is clearly stable by differentiation with respect to t, we deduce that $\mathscr{A}\mathscr{W} \in N(T)$. This implies that $\mathscr{A}\mathscr{N}(T) \subset \mathscr{N}(T) \subset \mathscr{L}_0^3 \times \mathscr{L}_{-1}^3$. Since $\mathscr{N}(T)$ is a finite dimensional closed subspace of $D(\mathscr{A})$, and stable by the action of the operator $\mathscr{A}$, it contains an eigenfunction of $\mathscr{A}$. Let us consider such an eigenfunction $(\phi_1^0, \phi_2^0, \phi_3^0, \phi_1^1, \phi_2^1, \phi_3^1) \in \mathscr{N}(T)$, associated to an eigenvalue $\nu \in \mathbb{C}$, so that we have

$$\begin{cases} \phi_1^1 & = \nu\phi_1^0, \\ \phi_2^1 & = \nu\phi_2^0, \\ \phi_3^1 & = \nu\phi_3^0, \\ -d_1\Delta_D\phi_1^0 & = \nu\phi_1^1, \\ -d_2\Delta_D\phi_2^0 - \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2}\phi_1^1 - \frac{a_1(-\Delta_D)^{-1}}{d_2}\phi_3^1 & = \nu\phi_2^1, \\ -d_2\Delta_D\phi_3^0 - a_2\phi_3^0 + \frac{a_1(-\Delta_D)^{-1}}{d_2}\phi_3^0 + \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2}\phi_1^0 - \frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2}\phi_1^0 + \phi_2^1 & = \nu\phi_3^1, \\ \left(\frac{\alpha_1 d_1^2}{(d_1-d_2)^2}\phi_1^0 + \phi_3^0\right)|_\omega & = 0. \end{cases} \tag{4.3.54}$$

Let us define a change of variables:

$$\begin{cases} \varphi_1 = d_1^2\Delta_D^2\phi_1^0, \\ \varphi_2 = \nu\phi_2^0 + \frac{\alpha_1 d_1^2}{d_2-d_1}\Delta_D\phi_1^0 + \frac{a_1(-\Delta_D)^{-1}}{d_2}(\frac{\alpha_1 d_1^2}{(d_1-d_2)^2}\phi_1^0 + \phi_3^0), \\ \varphi_3 = \frac{\alpha_1 d_1^2}{(d_1-d_2)^2}\phi_1^0 + \phi_3^0. \end{cases} \tag{4.3.55}$$

Remark 4.3.13. *We could make a link between the transform $\mathcal{S}'$ and (4.3.55). Formally, we are able to write*

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \mathcal{S}'(\nu, \Delta_D) \begin{pmatrix} \phi_1^0 \\ \phi_2^0 \\ \phi_3^0 \end{pmatrix}. \tag{4.3.56}$$

Here we use the notation $\mathcal{S}'(\nu, \Delta_D)$ to denote the transform replacing formally D_t by the eigenvalue ν (remind that $\mathcal{S}'$ involves only D_t and powers of Δ_D).

Then, we obtain a new system

$$\begin{cases} -d_1\Delta_D\varphi_1 & = \nu^2\varphi_1, \\ -d_2\Delta_D\varphi_2 + \alpha_1\varphi_1 - a_1\varphi_3 & = \nu^2\varphi_2, \\ -d_2\Delta_D\varphi_3 - a_2\varphi_3 + \varphi_2 & = \nu^2\varphi_3, \\ \varphi_3|_\omega & = 0. \end{cases} \tag{4.3.57}$$

Using the last equation of (4.3.57), we have

$$\varphi_2|_\omega = \left(\nu^2\varphi_3 + d_2\Delta_D\varphi_3 + a_2\varphi_3\right)|_\omega = 0.$$

Similarly, using the second equation of (4.3.57), we obtain $\varphi_1|_\omega = 0$. Since $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is the solution of the elliptic System (4.3.57) verifying $\varphi|_\omega = 0$, by usual unique continuation for elliptic systems, we obtain that $\varphi \equiv 0$ on Ω.

Let us now go back to the eigenvector $(\phi_1^0, \phi_2^0, \phi_3^0, \phi_1^1, \phi_2^1, \phi_3^1)$. The first line of (4.3.57) gives that $\alpha_1 d_1^2\Delta_D^2\phi_1^0 = 0$ on Ω. Since $\alpha_1 \neq 0$ by (4.3.3) and $\phi_1^0 = \Delta\phi_1^0 = 0$

on $\partial\Omega$, we deduce that $\phi_1^0 = 0$ on Ω. The first line of (4.3.54) also provides that $\phi_1^1 = 0$ on Ω. Working on the second line of (4.3.57) and then on the last line of (4.3.57), we obtain similarly that $\phi_2^0 = \phi_2^1 = \phi_3^0 = \phi_3^1 = 0$ on Ω, which concludes the proof.

□

4.3.3 The case $\tilde{\alpha}_2 \neq 0$

According to Lemma 4.2.2, given the initial condition

$$(\tilde{u}_1^{1,0}, \tilde{u}_1^{1,1}) \in H_\Omega^3(\Delta_D) \times H_\Omega^2(\Delta_D),$$

the solution $\tilde{u}_1^1$ to the first line of (4.3.5) satisfies

$$\tilde{u}_1^1 \in C^0([0,T], H_\Omega^3(\Delta_D)) \cap C^1([0,T], H_\Omega^2(\Delta_D)).$$

For technical reasons, we would like to work in symmetric spaces. We introduce a change of variables

$$\begin{cases} v_1^1 = D_t^2 \tilde{u}_1^1 + \dfrac{\tilde{\alpha}_2 d_2}{d_1 - d_2} \tilde{u}_2^2, \\ v_1^2 = D_t \tilde{u}_1^2, \\ v_2^2 = \tilde{u}_2^2. \end{cases}$$

with the inverse transform defined by

$$\begin{cases} \tilde{u}_1^1 = \frac{(-\Delta_D)^{-1}}{d_1} v_1^1 - \frac{\alpha_1 (-\Delta_D)^{-2}}{d_1 d_2} v_1^2 + \frac{(-\Delta_D)^{-1}}{d_1} (\alpha_1 - \frac{\alpha_2 d_1}{d_1 - d_2}) v_2^2, \\ \tilde{u}_1^2 = \frac{(-\Delta_D)^{-1}}{d_2} (D_t v_1^2 - v_2^2), \\ \tilde{u}_2^2 = v_2^2. \end{cases}$$

The exact controllability of System (4.3.5) is equivalent to the exact controllability in the state space $\mathscr{L}_1^3 \times \mathscr{L}_0^3$ of the system:

$$\begin{cases} \Box_1 v_1^1 + (\alpha_1 - \frac{\alpha_2 a_1 d_1 (-\Delta_D)^{-1}}{d_2 (d_1 - d_2)}) D_t v_1^2 - (\frac{a_2 \alpha_2 d_1}{d_1 - d_2} + \frac{\alpha_2 a_1 d_1 (-\Delta_D)^{-1}}{d_2 (d_1 - d_2)}) v_2^2 & = \frac{\alpha_2 d_1}{d_1 - d_2} f, \\ \Box_2 v_1^2 + D_t v_2^2 & = 0, \\ \Box_2 v_2^2 - \frac{a_1 (-\Delta_D)^{-1}}{d_2} D_t v_1^2 + (\frac{a_1 (-\Delta_D)^{-1}}{d_2} - a_2) v_2^2 & = f, \\ v_1^1|_\partial \Omega = 0, \; v_j^2|_\partial \Omega & = 0, \; j = 1, 2, \\ (v_1^1, v_1^2, v_2^2, \partial_t v_1^1, \partial_t v_1^2, \partial_t v_2^2)|_{t=0} & \in \mathscr{L}_1^3 \times \mathscr{L}_0^3. \end{cases} \tag{4.3.58}$$

It is equivalent to proving the following observability inequality: $\exists C > 0$ such that for any solutions of the adjoint system

$$\begin{cases} \Box_1 w_1^1 & = 0, \\ \Box_2 w_1^2 + (\alpha_1 - \frac{\alpha_2 a_1 d_1 (-\Delta_D)^{-1}}{d_2(d_1-d_2)}) D_t w_1^1 - \frac{a_1(-\Delta_D)^{-1}}{d_2} D_t w_2^2 & = 0, \\ \Box_2 w_2^2 + D_t w_1^2 + (\frac{a_1(-\Delta_D)^{-1}}{d_2} - a_2) w_2^2 - (\frac{a_2\alpha_2 d_1}{d_1-d_2} + \frac{\alpha_2 a_1 d_1 (-\Delta_D)^{-1}}{d_2(d_1-d_2)}) w_1^1 & = 0, \\ w_1^1|_{\partial\Omega} = 0,\ w_j^2|_{\partial\Omega} & = 0\, j = 1, 2, \\ (w_1^1, w_1^2, w_2^2)|_{t=0} = (w_1^{1,0}, w_1^{2,0}, w_2^{2,0}) & \in \mathscr{L}_0^3, \\ (\partial_t w_1^1, \partial_t w_1^2, \partial_t w_2^2)|_{t=0} = (w_1^{1,1}, w_1^{2,1}, w_2^{2,1}) & \in \mathscr{L}_{-1}^3. \end{cases} \tag{4.3.59}$$

we have the following observability inequality

$$C \int_0^T \int_\omega \left| \frac{\alpha_2 d_1}{d_1 - d_2} w_1^1 + w_2^2 \right|^2 dxdt \geq ||W(0)||^2_{\mathscr{L}_0^3 \times \mathscr{L}_{-1}^3}. \tag{4.3.60}$$

We follow the same procedure to prove the inequality (4.3.60) as we presented in Subsection 4.3.2. The proof is totally similar for the high frequency part. For the low frequency part, the same computations lead to consider a unique continuation property of the form

$$\begin{cases} -d_1 \Delta_D \varphi_1 & = \nu^2 \varphi_1, \\ -d_2 \Delta_D \varphi_2 + \alpha_1 \varphi_1 - a_1 \varphi_3 & = \nu^2 \varphi_2, \\ -d_2 \Delta_D \varphi_3 + \alpha_2 \varphi_1 + \varphi_2 - a_2 \varphi_3 & = \nu^2 \varphi_3, \\ \varphi_3|_\omega & = 0. \end{cases} \tag{4.3.61}$$

This system is very similar to (4.3.57). The main difference is that from the two last lines of (4.3.61), we only obtain for the moment that

$$\alpha_2 \varphi_1 + \varphi_2 = 0 \text{ on } \omega. \tag{4.3.62}$$

Using (4.3.62) with the first line of (4.3.61), we deduce that

$$d_1 \Delta_D \varphi_2 = -d_1 \alpha_2 \Delta_D \varphi_1 = \nu^2 \alpha_2 \varphi_1 \text{ on } \omega. \tag{4.3.63}$$

From (4.3.63) and the second line of (4.3.61), we deduce that

$$\left(d_1 \alpha_1 - \alpha_2 d_2 \nu^2\right) \varphi_1 - \nu^2 d_1 \varphi_2 = 0 \text{ on } \omega. \tag{4.3.64}$$

The unique solution of (4.3.62) and (4.3.64) is $\varphi_1 = \varphi_2 = 0$ on ω if

$$(\alpha_2)\left(-\nu^2 d_1\right) - 1\left(d_1 \alpha_1 - \alpha_2 d_2 \nu^2\right) \neq 0,$$

i.e.

$$\alpha_2 \nu^2 (d_1 - d_2) + d_1 \alpha_1 \neq 0.$$

The first line of (4.3.61) implies that there exists $\lambda \in \sigma(-\Delta_D)$ such that $\nu^2 = d_1\lambda$. Hence, $\varphi_1 = \varphi_2 = 0$ on ω if

$$\alpha_2\lambda\,(d_1 - d_2) + \alpha_1 \neq 0,$$

which is the case thanks to (4.3.3). Hence, we have $\varphi_1 = \varphi_2 = \varphi_3 = 0$ on ω, and we can then conclude exactly as in the previous case $\tilde{\alpha}_2 = 0$.

4.4 Proof of the sufficient part of Theorem 4.1.16

We organize this section a little bit differently from the previous section. We start by a modal problem to introduce the compatibility condition in this setting. We follow by a reformulation procedure of System (4.1.2). At last, we finish the proof of our main Theorem 4.1.16.

4.4.1 The modal case

Let $f \in L^2((0,T), L^2(\Omega))$. For a fixed $1 \leq s \leq n_2$, we consider the following system as a modal problem

$$\begin{cases} \Box_1 u_1^1 + \sum_{j=1}^{s} \alpha_j u_j^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_1^2 + u_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box_2 u_{n_2-1}^2 + u_{n_2}^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j} u_j^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ u_1^1 = 0, u_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega, 1 \leq j \leq n_2, \\ (u_1^1, u_1^2, \cdots, u_{n_2}^2)|_{t=0} & = (u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) & \text{in } \Omega, \\ (\partial_t u_1^1, \partial_t u_1^2, \cdots, \partial_t u_{n_2}^2)|_{t=0} & = (u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) & \text{in } \Omega. \end{cases} \tag{4.4.1}$$

In this section, we aim to prove the exact controllability of System (4.4.1) with the help of proper compatibility conditions. For this modal System (4.4.1), we have the following well-posedness property:

Proposition 4.4.1. *Assume that the initial conditions verify*

$$\begin{aligned} (u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) &\in H_\Omega^{n_2+3-s}(\Delta_D) \times H_\Omega^{n_2}(\Delta_D) \times \cdots \times H_\Omega^1(\Delta_D), \\ (u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) &\in H_\Omega^{n_2+2-s}(\Delta_D) \times H_\Omega^{n_2-1}(\Delta_D) \times \cdots \times H_\Omega^0(\Delta_D). \end{aligned}$$

Additionally, let us define $\tilde{U}^0$ *and* $\tilde{U}^1$ *by*

$$\tilde{U}^0 = (-d_1\Delta)^{n_2-s+1}u_1^{1,0} + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^{2,0}$$
$$+\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k} \frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^{2,0}, \tag{4.4.2}$$

and

$$\tilde{U}^1 = (-d_1\Delta)^{n_2-s+1}u_1^{1,1} + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^{2,1}$$
$$+\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k} \frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^{2,1}. \tag{4.4.3}$$

Assume that $\tilde{U}^0 \in H^1_\Omega(\Delta_D)$ *and* $\tilde{U}^1 \in H^0_\Omega(\Delta_D)$. *Then, the solution* $(u_1^1, u_1^2, \cdots, u_{n_2}^2)$ *satisfies*

$$\begin{aligned} &u_1^1 \in C^0([0,T], H_\Omega^{n_2+3-s}(\Delta)) \cap C^1([0,T], H_\Omega^{n_2+2-s}(\Delta)),\\ &u_j^2 \in C^0([0,T], H_\Omega^{n_2+1-j}(\Delta)) \cap C^1([0,T], H_\Omega^{n_2-j}(\Delta)), 1 \le j \le n_2. \end{aligned} \tag{4.4.4}$$

Furthermore, we have

$$\Bigg((-d_1\Delta)^{n_2-s+1}u_1^1 + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^2$$
$$+\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k} \frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^2\Bigg)$$
$$\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)). \tag{4.4.5}$$

Remark 4.4.2. *Let* $n_2 = 2, s = 1, \alpha_1 = 1$, *then* (4.4.5) *becomes the following condition:*

$$\Bigg((-d_1\Delta)^2u_1^1 + \sum_{k=0}^{1}\sum_{l=0}^{1-k} \binom{1-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{1-k-l}u_{1+l}^2$$
$$+\sum_{k=0}^{1}\sum_{l=0}^{1-k} \frac{d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{1-k}{l}(-d_2\Delta)^{1-k-l}u_{1+k+l}^2\Bigg)$$
$$\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)).$$

Simplifying the formula, we obtain that

$$\Big((-d_1\Delta)^2u_1^1 + \frac{d_1}{d_1-d_2}(-d_1\Delta)u_1^2 + \frac{d_1^2}{(d_1-d_2)^2}u_2^2\Big) \\ \in C^0([0,T],H^1_\Omega(\Delta_D)) \cap C^1([0,T],H^0_\Omega(\Delta_D)).$$

This is just the compatibility condition in the previous section.

Proof. As we have shown in the proof of Proposition 4.3.2, it is classical to obtain the regularity of the solutions given in (4.4.4), following Lemma 4.2.2. Now, we focus on the proof of the compatibility conditions (4.4.5), so we restrict to the case $s < n_2$ according to Remark 4.1.10. We perform the similar reformulation for the solutions of System (4.4.1):

$$\begin{cases} v_1^1 = D_t^{n_2+2-s}u_1^1, \\ v_1^2 = D_t^{n_2-1}u_1^2, \\ \vdots \\ v_{n_2}^2 = u_{n_2}^2. \end{cases} \tag{4.4.6}$$

The transform above is "invertible", and there are four different cases for the form of the inverse, that is, n_2 and $n_2 - s$ are both even or odd, n_2 is even while $n_2 - s$ is odd and the converse, that we do not detail here. We perform the same strategy as we have already shown in the proof of the Proposition 4.3.2. Thus, we obtain a system for $v_1^1, v_1^2, \cdots, v_{n_2}^2$ given by

$$\begin{cases} \Box_1 v_1^1 + \sum_{j=1}^{s}\alpha_j D_t^{n_2+2-s}u_j^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 v_1^2 + D_t v_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box_2 v_{n_2-1}^2 + D_t v_{n_2}^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 v_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j}u_j^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ v_1^1 = 0, v_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega, 1\le j\le n_2, \end{cases} \tag{4.4.7}$$

with initial conditions

$$\begin{aligned} (v_1^1, v_1^2, \cdots, v_{n_2}^2)|_{t=0} &= (v_1^{1,0}, v_1^{2,0}, \cdots, v_{n_2}^{2,0}), \\ (\partial_t v_1^1, \partial_t v_1^2, \cdots, \partial_t v_{n_2}^2)|_{t=0} &= (v_1^{1,1}, v_1^{2,1}, \cdots, v_{n_2}^{2,1}). \end{aligned} \tag{4.4.8}$$

We focus on the first equation. Let $y_0^1 = v_1^1 + \frac{\alpha_s d_2}{d_1-d_2} v_s^2$. Then, we obtain

$$\begin{aligned}
\square_1 y_0^1 &= \square_1 v_1^1 + \frac{\alpha_s d_2}{d_1 - d_2} \square_1 v_s^2 \\
&= -\sum_{j=1}^{s} \alpha_j D_t^{n_2-s+2} u_j^2 + \frac{\alpha_s d_2}{d_1 - d_2} \square_2 v_s^2 + \frac{\alpha_s d_2}{d_1 - d_2}(d_2 - d_1)\Delta v_s^2 \\
&= -\sum_{j=1}^{s} \alpha_j D_t^{n_2-s+2} u_j^2 - \frac{\alpha_s d_2}{d_1 - d_2} D_t v_{s+1}^2 - \alpha_s d_2 \Delta v_s^2.
\end{aligned}$$

Since v_s^2 satisfies the equation $\square_2 v_s^2 + D_t v_{s+1}^2 = 0$ by (4.4.1), we obtain that

$$\begin{aligned}
-\alpha_s D_t^{n_2-s+2} u_s^2 - \alpha_s d_2 \Delta v_s^2 &= -\alpha_s (D_t^2 v_s^2 + d_2 \Delta) v_s^2 \\
&= \alpha_s \square_2 v_s^2 \\
&= -\alpha_s D_t v_{s+1}^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\square_1 y_0^1 &= -\sum_{j=1}^{s-1} \alpha_j D_t^{n_2-s+2} u_j^2 - \frac{\alpha_s d_2}{d_1 - d_2} D_t v_{s+1}^2 - \alpha_s D_t v_{s+1}^2 \\
&= -\sum_{j=1}^{s-1} \alpha_j D_t^{n_2-s+2} u_j^2 - \alpha_s (\frac{d_2}{d_1 - d_2} + 1) D_t v_{s+1}^2 \\
&= -\sum_{j=1}^{s-1} \alpha_j D_t^{n_2-s+2} u_j^2 - \frac{\alpha_s d_1}{d_1 - d_2} D_t v_{s+1}^2.
\end{aligned}$$

As a consequence, using the definition $v_{s-1}^2 = D_t^{n_2-s+1} u_{s-1}^2$, we know that y_0^1 satisfies the equation

$$\square_1 y_0^1 + \sum_{j=1}^{s-2} \alpha_j D_t^{n_2-s+2} u_j^2 + \frac{\alpha_s d_1}{d_1 - d_2} D_t v_{s+1}^2 + \alpha_{s-1} D_t v_{s-1}^2 = 0. \tag{4.4.9}$$

Define by induction

$$y_j^1 = D_t y_{j-1}^1 + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1 - d_2)^{j+1-k}} v_{s+j-2k}^2, 1 \le j \le n_2 - s - 1. \tag{4.4.10}$$

Let $\alpha_j = 0$ for $j \in \mathbb{Z} \setminus \{1, 2, \cdots, s\}$. We have the following lemmas, which are proved afterwards.

Lemma 4.4.3. *y_j^1 $(1 \leq j \leq n_2 - s - 1)$ satisfies the equation*

$$\Box_1 y_j^1 + \sum_{k=-\infty}^{s-2-j} \alpha_k D_t^{n_2-s+j+2} u_k^2 + \sum_{k=0}^{j+1} \frac{\alpha_{s-k} d_1^{j+1-k}}{(d_1 - d_2)^{j+1-k}} D_t v_{s+j+1-2k}^2 = 0. \tag{4.4.11}$$

Remark 4.4.4. *$\sum_{k=-\infty}^{s-2-l} \alpha_k D_t^{n_2-s+2} u_k^2$ is a sum of finite terms, since for $k \leq 0$, $\alpha_k \equiv 0$.*

Let $y_{comp} = D_t y_{n_2-s-1}^1 + \sum_{k=0}^{n_2-s} \frac{\alpha_{s-k} d_2 d_1^{n_2-s-k}}{(d_1-d_2)^{n_2-s+1-k}} v_{n_2-2k}^2$.

Lemma 4.4.5. *y_{comp} satisfies the equation*

$$\begin{aligned} \Box_1 y_{comp} = &- \sum_{k=1}^{n_2-s+1} \frac{\alpha_{s-k} d_1^{n_2-s+1-k}}{(d_1 - d_2)^{n_2-s+1-k}} D_t v_{n_2+1-2k}^2 - \sum_{k=-\infty}^{2s-2-n_2} \alpha_k D_t^{2n_2-2s+2} u_k^2 \\ &+ \sum_{k=1}^{n_2} \frac{a_{n_2+1-k} \alpha_s d_1^{n_2+1-s}}{(d_1 - d_2)^{n_2+1-s}} u_k^2 + \frac{\alpha_s d_1^{n_2+1-s}}{(d_1 - d_2)^{n_2+1-s}} f. \end{aligned} \tag{4.4.12}$$

Lemma 4.4.6. *For y_{comp}, we have*

$$\begin{aligned} y_{comp} = &(-d_1 \Delta)^{n_2-s+1} u_1^1 \\ &+ \sum_{k=0}^{n_2-s} \sum_{j=1}^{s} \sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2 - s - k}{l} (-d_1 \Delta)^k (-d_2 \Delta)^{n_2-s-k-l} u_{j+l}^2 \\ &+ \sum_{j=1}^{s} \sum_{k=0}^{n_2-2s+j} \sum_{l=0}^{n_2-s-k} \frac{\alpha_{s-k} d_2 d_1^k}{(d_1 - d_2)^{k+1}} \binom{n_2 - s - k}{l} (-d_2 \Delta)^{n_2-s-k-l} u_{j+k+l}^2. \end{aligned} \tag{4.4.13}$$

Assume for the moment that these Lemmas are true and let us complete the proof of Proposition 4.4.1. Define

$$\begin{aligned} F = &- \sum_{k=1}^{n_2-s+1} \frac{\alpha_{s-k} d_1^{n_2-s+1-k}}{(d_1 - d_2)^{n_2-s+1-k}} D_t v_{s+j+1-2k}^2 - \sum_{k=-\infty}^{2s-2-n_2} \alpha_k D_t^{2n_2-2s+2} u_k^2 \\ &+ \sum_{k=1}^{n_2} \frac{a_{n_2+1-k} \alpha_s d_1^{n_2+1-s}}{(d_1 - d_2)^{n_2+1-s}} u_k^2 + \frac{\alpha_s d_1^{n_2+1-s}}{(d_1 - d_2)^{n_2+1-s}} f. \end{aligned} \tag{4.4.14}$$

Since

$$u_k^2 \in C^0([0,T], H_\Omega^{n_2+1-k}(\Delta_D)) \cap C^1([0,T], H_\Omega^{n_2-k}(\Delta_D)),$$

we know that

$$D_t^{2n_2-2s+2} u_k^2 \in L^1((0,T), H_\Omega^0(\Delta_D)), \; k \leq 2s - 2 - n_2,$$

which implies that $F \in L^1((0,T), H^0_\Omega(\Delta_D))$. Now, we remark that by (4.4.4) and (4.4.5), y_{comp} satisfies

$$\begin{aligned} y_{comp}|_{t=0} &= \tilde{U}^0 \in H^1_\Omega(\Delta_D), \\ \partial_t y_{comp}|_{t=0} &= \tilde{U}^1 \in H^0_\Omega(\Delta_D). \end{aligned}$$

Consequently, from (4.4.12), (4.4.14) and the fact that $F \in L^1((0,T), H^0_\Omega(\Delta_D))$,we conclude that $y_{comp} \in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D))$. $\square$

It only remains to prove Lemma 4.4.3, Lemma 4.4.5 and Lemma 4.4.6.

Proof of Lemma 4.4.3 *and Lemma* 4.4.5. We prove these lemmas by induction. For y^1_0, according to (4.4.9), we know that y^1_0 satisfies (4.4.11) for $j=1$. Assume that for $l<j$, y^1_l satisfies (4.4.11). Thus, using the definition of y^1_j and the equation for y^1_{j-1}, we know that y^1_j satisfies the following equation

$$\begin{aligned} \Box_1 y^1_j &= D_t \Box_1 y^1_{j-1} + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} \Box_1 v^2_{s+j-2k} \\ &= -\sum_{k=-\infty}^{s-1-j} \alpha_k D_t^{n_2-s+j+2} u^2_k - \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} D_t^2 v^2_{s+j-2k} \\ &+ \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} \Box_2 v^2_{s+j-2k} + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} (d_2-d_1)\Delta v^2_{s+j-2k}. \end{aligned}$$

By simple observation, we know that

$$\begin{aligned} &-\sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} D_t^2 v^2_{s+j-2k} + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} (d_2-d_1)\Delta v^2_{s+j-2k} \\ =&\sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} \partial_t^2 v^2_{s+j-2k} + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} (-d_2\Delta) v^2_{s+j-2k} \\ =&\sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} \Box_2 v^2_{s+j-2k}. \end{aligned}$$

Therefore, we simplify the equation for y^1_j,

$$\begin{aligned} \Box_1 y^1_j &= -\sum_{k=-\infty}^{s-1-j} \alpha_k D_t^{n_2-s+j+2} u^2_k + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} \left(\frac{d_2}{d_1-d_2}+1\right) \Box_2 v^2_{s+j-2k} \\ &= -\sum_{k=-\infty}^{s-1-j} \alpha_k D_t^{n_2-s+j+2} u^2_k + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k+1}}{(d_1-d_2)^{j-k+1}} \Box_2 v^2_{s+j-2k}. \end{aligned}$$

Using the equation $\Box_2 v^2_{s+j-2k} = -D_t v^2_{s+1+j-2k}$ coming from (4.4.7), we obtain

$$\Box_1 y^1_j = -\sum_{k=-\infty}^{s-1-j} \alpha_k D_t^{n_2-s+j+2} u^2_k - \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k+1}}{(d_1-d_2)^{j-k+1}} D_t v^2_{s+j-2k+1}.$$

Now we look at the term $\alpha_{s-1-j} D_t^{n_2-s+j+2} u^2_{s-1-j}$. If $j \le s-1$, we obtain

$$\alpha_{s-1-j} D_t^{n_2-s+j+2} u^2_{s-1-j} = \alpha_{s-1-j} D_t v^2_{s-1-j};$$

if $j > s-1$, $\alpha_{s-1-j} = 0$. Hence, we have

$$\Box_1 y^1_j = -\sum_{k=-\infty}^{s-2-j} \alpha_k D_t^{n_2-s+j+2} u^2_k - \sum_{k=0}^{j+1} \frac{\alpha_{s-k} d_1^{j-k+1}}{(d_1-d_2)^{j-k+1}} D_t v^2_{s+j-2k+1}.$$

By induction, this implies that $y^1_j (1 \le j \le n_2 - s - 1)$ satisfies the equation

$$\Box_1 y^1_j + \sum_{k=-\infty}^{s-2-j} \alpha_k D_t^{n_2-s+j+2} u^2_k + \sum_{k=0}^{j+1} \frac{\alpha_{s-k} d_1^{j+1-k}}{(d_1-d_2)^{j+1-k}} D_t v^2_{s+j+1-2k} = 0. \tag{4.4.15}$$

Using the definition of y_{comp}, we obtain

$$\Box_1 y_{comp} = D_t \Box_1 y^1_{n_2-s-1} + \sum_{k=0}^{n_2-s} \frac{\alpha_{s-k} d_2 d_1^{n_2-s-k}}{(d_1-d_2)^{n_2-s+1-k}} \Box_1 v^2_{n_2-2k}.$$

Following the same procedure, we have the following equation

$$\Box_1 y_{comp} = -\sum_{k=-\infty}^{2s-1-n_2} \alpha_k D_t^{2n_2-2s+2} u^2_k + \sum_{k=0}^{n_2-s} \frac{\alpha_{s-k} d_1^{n_2-s-k+1}}{(d_1-d_2)^{n_2-s-k+1}} \Box_2 v^2_{n_2-2k}.$$

Using the equation $\Box_2 v^2_{n_2} = \sum_{k=1}^{n_2} a_{n_2+1-k} u^2_k + f$ coming from (4.4.7), we obtain

$$\begin{aligned}\Box_1 y_{comp} = &-\sum_{k=-\infty}^{2s-1-n_2} \alpha_k D_t^{2n_2-2s+2} u^2_k + \sum_{k=1}^{n_2-s} \frac{\alpha_{s-k} d_1^{n_2-s-k+1}}{(d_1-d_2)^{n_2-s-k+1}} D_t v^2_{n_2-2k+1} \\ &+ \sum_{k=1}^{n_2} \frac{a_{n_2+1-k} \alpha_s d_1^{n_2-s+1}}{(d_1-d_2)^{n_2-s+1}} u^2_k + \frac{\alpha_s d_1^{n_2-s+1}}{(d_1-d_2)^{n_2-s+1}} f.\end{aligned}$$

Now look at the term $\alpha_{2s-1-n_2} D_t^{2n_2-2s+2} u^2_{2s-1-n_2}$. If $2s-1-n_2 \le 0$, we know that $\alpha_{2s-1-n_2} \equiv 0$. Otherwise, we know that $D_t^{2n_2-2s+2} u^2_{2s-1-n_2} = D_t v^2_{2s-1-n_2}$.

Consequently, we obtain the equation for y_{comp}:

$$\Box_1 y_{comp} = -\sum_{k=1}^{n_2-s+1} \frac{\alpha_{s-k} d_1^{n_2-s+1-k}}{(d_1-d_2)^{n_2-s+1-k}} D_t v^2_{n_2+1-2k} - \sum_{k=-\infty}^{2s-2-n_2} \alpha_k D_t^{2n_2-2s+2} u_k^2$$
$$+ \sum_{k=1}^{n_2} \frac{a_{n_2+1-k}\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} u_k^2 + \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} f,$$

which is exactly the equation (4.4.12). $\Box$

Proof of Lemma 4.4.6. Recall the definition of y_{comp},

$$y_{comp} = D_t y^1_{n_2-s-1} + \sum_{k=0}^{n_2-s} \frac{\alpha_{s-k} d_2 d_1^{n_2-s-k}}{(d_1-d_2)^{n_2-s+1-k}} v^2_{n_2-2k},$$

and the definition of $y_j^1 (1 \le j \le n_2 - s - 1)$,

$$y_j^1 = D_t y^1_{j-1} + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} v^2_{s+j-2k}.$$

Therefore, by iteration, we have the following expression for y_{comp}

$$y_{comp} = D_t^{n_2-s} y_0^1 + \sum_{j=1}^{n_2-s} \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} D_t^{n_2-s-j} v^2_{s+j-2k}. \tag{4.4.16}$$

Using the definitions of $y_0^1 = v_1^1 + \frac{\alpha_s d_2}{d_1-d_2} v_s^2$ and $v_j^2 = D_t^{n_2+1-j} u_j^2, 1 \le j \le n_2$ given in (4.4.6), we simplify the formula above:

$$y_{comp} = D_t^{2n_2-2s+2} u_1^1 + \sum_{j=0}^{n_2-s} \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} D_t^{2n_2-2s-2j+2k} u^2_{s+j-2k}.$$

According to the equation $D_t^2 u_1^1 = -d_1 \Delta u_1^1 + \sum_{j=1}^{s} \alpha_j u_j^2$ coming from (4.4.1), we obtain

$$\begin{aligned} y_{comp} &= D_t^{2n_2-2s}(-d_1 \Delta u_1^1 \\ &\quad + \sum_{j=1}^{s} \alpha_j u_j^2) + \sum_{j=0}^{n_2-s} \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} D_t^{2n_2-2s-2j+2k} u^2_{s+j-2k} \\ &= (-d_1\Delta) D_t^{n_2-s} u_1^1 + \sum_{j=1}^{s} \alpha_j D_t^{n_2-s} u_j^2 \\ &\quad + \sum_{j=0}^{n_2-s} \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} D_t^{2n_2-2s-2j+2k} u^2_{s+j-2k}. \end{aligned}$$

By iteration, we are able to obtain that

$$
\begin{aligned}
y_{comp} = (-d_1\Delta)^{n_2-s+1}u_1^1 + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\alpha_j(-d_1\Delta)^k D_t^{2n_2-2s-2k}u_j^2 \\
+ \sum_{j=0}^{n_2-s}\sum_{k=0}^{j}\frac{\alpha_{s-k}d_2d_1^{j-k}}{(d_1-d_2)^{j+1-k}}D_t^{2n_2-2s-2j+2k}u_{s+j-2k}^2.
\end{aligned}
$$

Now we introduce the following lemma to describe the term $D_t^{2k}u_j^2$.

Lemma 4.4.7. *Let u_j^2 be solutions to the system* (4.4.1)*. If $k+j \le n_2$, we have*

$$
D_t^{2k}u_j^2 = \sum_{l=0}^{k}\binom{k}{l}(-d_2\Delta)^l u_{j+k-l}^2. \tag{4.4.17}
$$

We shall prove this lemma in Appendix B. Now, we use this lemma to simplify the formula of y_{comp}. In the term $\sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\alpha_j(-d_1\Delta)^k D_t^{2n_2-2s-2k}u_j^2$, since $j \le s$ and $k \ge 0$, we know that $n_2 - s - k + j \le n_2 - k \le n_2$. Thus, according to Lemma 4.4.7, we obtain

$$
D_t^{2n_2-2s-2k}u_j^2 = \sum_{l=0}^{n_2-s-k}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^2. \tag{4.4.18}
$$

On the other hand, in the term $\sum_{j=0}^{n_2-s}\sum_{k=0}^{j}\frac{\alpha_{s-k}d_2d_1^{j-k}}{(d_1-d_2)^{j+1-k}}D_t^{2n_2-2s-2j+2k}u_{s+j-2k}^2$, since $k \ge 0$, we know that $(s+j-2k)+(n_2-s-j+k) = n_2-k \le n_2$. Therefore, according to Lemma 4.4.7, we obtain

$$
D_t^{2n_2-2s-2j+2k}u_{s+j-2k}^2 = \sum_{l=0}^{n_2-s-j+k}\binom{n_2-s-j+k}{l}(-d_2\Delta)^{n_2-s-j+k-l}u_{s+j-2k+l}^2. \tag{4.4.19}
$$

As a consequence, we obtain that

$$
\begin{aligned}
y_{comp} &= (-d_1\Delta)^{n_2-s+1}u_1^1 \\
&+ \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k}\alpha_j\binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^2 \\
&+ \sum_{j=0}^{n_2-s}\sum_{k=0}^{j}\sum_{l=0}^{n_2-s-j+k}\frac{\alpha_{s-k}d_2d_1^{j-k}}{(d_1-d_2)^{j+1-k}}\binom{n_2-s-j+k}{l}(-d_2\Delta)^{n_2-s-j+k-l}u_{s+j-2k+l}^2.
\end{aligned} \tag{4.4.20}
$$

For the last term in the formula above, since $\alpha_{s-k} = 0$ for $k \geq s$, we know that

$$\begin{aligned}
&\sum_{j=0}^{n_2-s}\sum_{k=0}^{j}\sum_{l=0}^{n_2-s-j+k} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} \binom{n_2-s-j+k}{l} (-d_2\Delta)^{n_2-s-j+k-l} u^2_{s+j-2k+l} \\
=&\sum_{k=0}^{s-1}\sum_{j=k}^{n_2-s}\sum_{l=0}^{n_2-s-j+k} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} \binom{n_2-s-j+k}{l} (-d_2\Delta)^{n_2-s-j+k-l} u^2_{s+j-2k+l} \\
=&\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k} \frac{\alpha_j d_2 d_1^{k}}{(d_1-d_2)^{k+1}} \binom{n_2-s-k}{l} (-d_2\Delta)^{n_2-s-k-l} u^2_{j+k+l}.
\end{aligned}$$

The last equality holds after a change of the sum index. Therefore, we obtain the form for y_{comp}

$$\begin{aligned}
y_{comp} = &(-d_1\Delta)^{n_2-s+1} u_1^1 \\
&+ \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l} (-d_1\Delta)^k (-d_2\Delta)^{n_2-s-k-l} u^2_{j+l} \\
&+ \sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k} \frac{\alpha_j d_2 d_1^{k}}{(d_1-d_2)^{k+1}} \binom{n_2-s-k}{l} (-d_2\Delta)^{n_2-s-k-l} u^2_{j+k+l}.
\end{aligned} \tag{4.4.21}$$

□

We also have the similar theorem as we proved in the previous section:

Theorem 4.4.8. *Given $T > 0$, suppose that:*

1. *(ω, T, p_{d_i}) satisfies GCC, $i = 1, 2$.*
2. *Ω has no infinite order of tangential contact with the boundary.*

Then System (4.4.1) *is exactly controllable in $\mathcal{H}_1^s \times \mathcal{H}_0^s$.*

As before, proving Theorem 4.4.8 is equivalent to proving the exact controllability of the following system:

$$\begin{cases}
\Box_1 v_1^1 + R(v_1^2, \cdots, v_{n_2}^2) & = \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} f \mathbb{1}_\omega \quad \text{in } (0,T)\times\Omega, \\
\Box_2 v_1^2 + D_t v_2^2 & = 0 \quad \text{in } (0,T)\times\Omega, \\
\vdots & \\
\Box_2 v_{n_2}^2 - \sum_{k=1}^{n_2} a_{n_2+1-k} \mathcal{S}_k^{-1}(v_k^2, \cdots, v_{n_2}^2) & = f\mathbb{1}_\omega \quad \text{in } (0,T)\times\Omega, \\
v_1^1 = 0, v_1^2 = \cdots = v_{n_2}^2 & = 0 \quad \text{on } (0,T)\times\partial\Omega,, \\
(v_1^1, v_1^2, \cdots, v_{n_2}^2)|_{t=0} \in \mathscr{L}_1^{n_2+1} & \\
(\partial_t v_1^1, \partial_t v_1^2, \cdots, \partial_t v_{n_2}^2)|_{t=0} \in \mathscr{L}_0^{n_2+1}, &
\end{cases} \tag{4.4.22}$$

with

$$\begin{aligned}
R(v_1^2, \cdots, v_{n_2}^2) = & \sum_{k=1}^{n_2-s+1} \frac{\alpha_{s-k} d_1^{n_2-s+1-k}}{(d_1-d_2)^{n_2-s+1-k}} D_t v_{s+j+1-2k}^2 \\
& + \sum_{k=-\infty}^{2s-2-n_2} \alpha_k D_t^{2n_2-2s+2} \mathcal{S}_k^{-1}(v_k^2, \cdots, v_{n_2}^2) \\
& + \sum_{k=1}^{n_2} \frac{a_{n_2+1-k} \alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} \mathcal{S}_k^{-1}(v_k^2, \cdots, v_{n_2}^2).
\end{aligned}$$

Here we use the transform $\mathcal{S}$ given by

$$\mathcal{S}\begin{pmatrix} u_1^1 \\ u_1^2 \\ \vdots \\ u_{n_2}^2 \end{pmatrix} = \begin{pmatrix} v_1^1 \\ v_1^2 \\ \vdots \\ v_{n_2}^2 \end{pmatrix},$$

where

$$\begin{cases} v_1^1 = y_{comp} \\ v_1^2 = D_t^{n_2-1} u_1^2, \\ \vdots \\ v_{n_2}^2 = u_{n_2}^2, \end{cases} \tag{4.4.23}$$

with

$$\begin{aligned}
y_{comp} = & (-d_1\Delta)^{n_2-s+1} u_1^1 \\
& + \sum_{k=0}^{n_2-s} \sum_{j=1}^{s} \sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l} (-d_1\Delta)^k (-d_2\Delta)^{n_2-s-k-l} u_{j+l}^2 \\
& + \sum_{j=1}^{s} \sum_{k=0}^{n_2-2s+j} \sum_{l=0}^{n_2-s-k} \frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}} \binom{n_2-s-k}{l} (-d_2\Delta)^{n_2-s-k-l} u_{j+k+l}^2.
\end{aligned}$$

Remark that Proposition 4.3.2 together with (4.4.23) ensures that

$$(v_1^1, v_1^2, \ldots, v_{n_2}^2) \in C^0([0,T], \mathscr{L}_1^{n_2+1}) \cap C^1([0,T], \mathscr{L}_0^{n_2+1}).$$

We use $\mathcal{S}^{-1}$ to denote the inverse transform given by

$$\begin{cases} u_1^1 = \mathcal{S}_0^{-1}(v_1^1, v_1^2, \cdots, v_{n_2}^2), \\ u_1^2 = \mathcal{S}_1^{-1}(v_1^2, \cdots, v_{n_2}^2), \\ \vdots \\ u_{n_2-j}^2 = \mathcal{S}_j^{-1}(v_{n_2-j}^2, \cdots, v_{n_2}^2), 0 \le j \le n_2 - 1, \\ \vdots \\ u_{n_2}^2 = \mathcal{S}_{n_2}^{-1}(v_{n_2}^2) = v_{n_2}^2. \end{cases} \tag{4.4.24}$$

Then, we treat exactly the same way as we did in the proof of Proposition 4.3.2 to obtain the form of the inverse transform of $\mathcal{S}$. There are two different cases. For $n_2 = 2k+1$, which is an odd integer, we are able to obtain that

$$\begin{cases} u^2_{2k+1} & = v^2_{2k+1}, \\ u^2_{2k} & = (-d_2\Delta_D)^{-1} D_t v^2_{2k} + T(2k, 2k+1)(-d_2\Delta_D)^{-1} v^2_{2k+1}, \\ \vdots & \\ u^2_1 & = (-d_2\Delta_D)^{-k} v^2_1 + T(1,2)(-d_2\Delta_D)^{-k-1} D_t v^2_2 \cdots \\ & \quad + T(1, 2k+1)(-d_2\Delta_D)^{-2k} v^2_{2k+1}. \end{cases} \tag{4.4.25}$$

It is similar for the even integer $n_2 = 2k$:

$$\begin{cases} u^2_{2k} & = v^2_{2k}, \\ u^2_{2k-1} & = (-d_2\Delta_D)^{-1} D_t v^2_{2k-1} + T(2k-1, 2k)(-d_2\Delta_D)^{-1} v^2_{2k}, \\ \vdots & \\ u^2_1 & = (-d_2\Delta_D)^{-k} D_t v^2_1 + T(1,2)(-d_2\Delta_D)^{-k} v^2_2 \cdots + T(1, 2k)(-d_2\Delta_D)^{1-2k} v^2_{2k}. \end{cases} \tag{4.4.26}$$

Here the coefficients $\{T(i,j)\}_{1\le i<j\le n}$ are uniquely determined by System (4.4.1), but their exact value is not really important.

Remark 4.4.9. *As explained in Remark* 4.3.3*, we are able to rewrite the system* (4.4.22) *as follows:*

$$(\partial_t^2 - \mathcal{S}D\mathcal{S}^{-1}\Delta + \mathcal{S}A\mathcal{S}^{-1})V = \mathcal{S}\hat{b}f,$$

and we have

$$\mathcal{S}\hat{b}f = \begin{pmatrix} \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} f, \\ 0, \\ \vdots \\ 0, \\ f \end{pmatrix}.$$

Moreover, we could notice that both $\mathcal{S}$ and $\mathcal{S}^{-1}$ only involve D_t and $(-\Delta_D)^k, k \in \mathbb{Z}$.

According to the Hilbert Uniqueness Method, we only need to prove the observability inequality

$$C\int_0^T \int_\omega \left| \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} w^1_1 + w^2_{n_2} \right|^2 dxdt \ge ||W(0)||^2_{\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}}. \tag{4.4.27}$$

for any solution of the adjoint system:

$$\begin{cases} \Box_1 w_1^1 & = 0 \quad \text{in } (0,T)\times\Omega, \\ \Box_2 w_1^2 + \Lambda_1 w_{n_2}^2 + \tilde{\Lambda}_1 w_1^1 & = 0 \quad \text{in } (0,T)\times\Omega, \\ \Box_2 w_2^2 + D_t w_1^2 + \Lambda_2 w_{n_2}^2 + \tilde{\Lambda}_2 w_1^1 & = 0 \quad \text{in } (0,T)\times\Omega, \\ \vdots & \\ \Box_2 w_{n_2}^2 + D_t w_{n_2-1}^2 + \Lambda_{n_2} w_{n_2}^2 + \tilde{\Lambda}_{n_2} w_1^1 & = 0 \quad \text{in } (0,T)\times\Omega, \\ w_1^1 = 0, w_1^2 = \cdots = w_{n_2}^2 & = 0 \quad \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{4.4.28}$$

with initial conditions

$$\begin{aligned} (w_1^1, w_1^2, \cdots, w_{n_2}^2)|_{t=0} &\in (L^2(\Omega))^{n_2+1} = \mathscr{L}_0^{n_2+1} \\ (\partial_t w_1^1, \partial_t w_1^2, \cdots, \partial_t w_{n_2}^2)|_{t=0} &\in (H_\Omega^{-1}(\Delta_D))^{n_2+1} = \mathscr{L}_{-1}^{n_2+1}, \end{aligned}$$

where the operators $(\Lambda_j)_{1\le j\le n_2}$ and $(\tilde{\Lambda}_j)_{1\le j\le n_2}$ are uniquely determined by the transform (4.4.23) and additionally are bounded operators in $L^2(\Omega)$. As usual, we divide the proof of the observability inequality (4.4.27) into two steps.

Remark 4.4.10. *We are able to rewrite the adjoint system* (4.4.28) *as follows*

$$(\partial_t^2 - (\mathcal{S}')^{-1} D \mathcal{S}' \Delta + (\mathcal{S}')^{-1} A^* \mathcal{S}') W = 0.$$

Here the transform $\mathcal{S}'$ denotes the invertible transform between the adjoint systems. Moreover, we could notice that both $\mathcal{S}'$ and $(\mathcal{S}')^{-1}$ only involve D_t and $(-\Delta_D)^k, k \in \mathbb{Z}$.

Step 1: establish a relaxed observability inequality.

First, we can establish a relaxed observability inequality for the adjoint System (4.4.28).

Proposition 4.4.11. *For solutions of System* (4.4.28)*, there exists a constant $C > 0$ such that*

$$\begin{aligned} &||W(0)||^2_{\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}} \\ &\le C\left(\int_0^T\int_\omega \left|\frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} w_1^1 + w_{n_2}^2\right|^2 dxdt + ||W(0)||^2_{\mathscr{L}_{-1}^{n_2+1}\times\mathscr{L}_{-2}^{n_2+1}}\right). \end{aligned} \tag{4.4.29}$$

Proof. We argue by contradiction. Suppose that the observability inequality (4.4.29) is not satisfied. Thus, there exists a sequence $(W^k)_{k\in\mathbb{N}}$ the solutions of System

(4.4.28) such that

$$||W^k(0)||^2_{\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}} = 1, \tag{4.4.30}$$

$$\int_0^T\int_\omega \left|\frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} w_1^{1,k} + w_{n_2}^{2,k}\right|^2 dxdt \to 0 \text{ as } k\to\infty, \tag{4.4.31}$$

$$||W^k(0)||^2_{\mathscr{L}_{-1}^{n_2+1}\times\mathscr{L}_{-2}^{n_2+1}} \to 0 \text{ as } k\to\infty. \tag{4.4.32}$$

By the continuity of the solution with respect to the initial data of System (4.3.27), we know that the sequence $(W^k)_{k\in\mathbb{N}}$ is bounded in $(L^2((0,T)\times\Omega))^{n_2+1}$ and moreover, $W^k \rightharpoonup 0$ in $(L^2((0,T)\times\Omega))^{n_2+1}$. We have W^k satisfying the following system

$$\begin{cases} \Box w_1^{1,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega, \\ \Box w_1^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega, \\ \Box w_2^{2,k} + D_t w_1^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box w_{n_2}^{2,k} + D_t w_{n_2-1}^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega. \end{cases} \tag{4.4.33}$$

Hence, we obtain two microlocal defect measures $\underline{\mu}_1 \in \underline{\mathcal{M}}^+$ and $\underline{\mu}_2 \in \underline{\mathcal{M}}^+$ associated with $(w_1^{1,k})_{k\in\mathbb{N}}$ and $(W^{2,k})_{k\in\mathbb{N}}$ respectively. From the definition in Proposition 4.2.6, we know that

$$\forall A\in\underline{\mathcal{A}},\quad \langle\underline{\mu}_1,\sigma(A)\rangle = \lim_{k\to\infty}(A\underline{w}_1^{1,k},\underline{w}_1^{1,k})_{L^2},$$
$$\langle\underline{\mu}_2(i,j),\sigma(A)\rangle = \lim_{k\to\infty}(A\underline{w}_i^{2,k},\underline{w}_j^{2,k})_{L^2}, 1\le i,j\le 2.$$

Here $\underline{\mu}_2 = (\underline{\mu}_2(i,j))_{1\le i,j\le n_2}$ is the matrix measure associated with the sequence $(W^{2,k})_{k\in\mathbb{N}} = (w_1^{2,k},\cdots,w_{n_2}^{2,k})_{k\in\mathbb{N}}$ and moreover, $\underline{w}_1^{1,k}$ and $\underline{w}_i^{2,k}$ is the extension by 0 across the boundary of $\Omega(1\le i\le n_2)$. As we already presented in the Subsection 4.3.2, the two measures are mutually singular in $(0,T)\times\Omega$. Then provided with

$$\int_0^T\int_\omega \left|\frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} w_1^{1,k} + w_{n_2}^{2,k}\right|^2 dxdt \to 0 \text{ as } k\to\infty,$$

we obtain that for $\chi\in C_0^\infty((0,T)\times\omega)$

$$\langle\frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}\chi w_1^{1,k}, \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}\chi w_1^{1,k}\rangle \to 0,$$
$$\langle\chi w_{n_2}^{2,k},\chi w_{n_2}^{2,k}\rangle \to 0, \qquad \text{as } k\to\infty.$$

Thus, we know that

$$\underline{\mu}_1|_{(0,T)\times\omega} = 0, \text{ and } \underline{\mu}_2(n_2,n_2)|_{(0,T)\times\omega} = 0. \tag{4.4.34}$$

For $\underline{\mu}_1$, since $\underline{\mu}_1$ is invariant along the along the general bicharacteristics of p_{d_1}, combining with GCC, we know that $\underline{\mu}_1 \equiv 0$. For $\underline{\mu}_2$, we consider the other definition of the microlocal defect measure. From Proposition 4.2.8, we know that there exists a measure $\mu_2 \in \mathcal{M}^+$ such that

$$\forall A \in \mathcal{A}, \quad \langle \mu_2, \kappa(\sigma(A)) \rangle = \lim_{k\to\infty} (AW^{2,k}, W^{2,k})_{L^2}. \tag{4.4.35}$$

Here $\mu_2 = (\mu_2(i,j))_{1\le i,j\le n_2}$ is a matrix measure. Since $\underline{\mu}_2|_{Char(p_{d_2})} = \mu_2$ μ_2-almost surely, we obtain that $\mu_2(n_2, n_2)|_{(0,T)\times\omega} = 0$. As we already presented in the Subsection 4.3.2, we would like to use Lemma 4.2.10. So we adapt this lemma under our setting here.

Lemma 4.4.12. *Assume that μ_2 is the corresponding microlocal defect measure defined by*

$$\forall A \in \mathcal{A}, \quad \langle \mu_2, \kappa(\sigma(A)) \rangle = \lim_{k\to\infty} (AW^{2,k}, W^{2,k})_{L^2}. \tag{4.4.36}$$

for the sequence $W^{2,k} = (w_1^{2,k}, \cdots, w_{n_2}^{2,k})_{k\in\mathbb{N}}$ which satisfies the following system:

$$\begin{cases} \Box w_1^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega, \\ \Box w_2^{2,k} + D_t w_1^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box w_{n_2}^{2,k} + D_t w_{n_2-1}^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega. \end{cases} \tag{4.4.37}$$

If we denote the general bicharacteristic by $s \mapsto \gamma(s)$, then along $\gamma(s)$ there exists a continuous function $s \mapsto M(s)$ such that M satisfies the differential equation:

$$\frac{d}{ds}(M(s)) = iE(\tau)M(s), M(0) = Id,$$

and μ_2 is invariant along the flow associated with M, which means that

$$\frac{d}{ds}(M^*\mu_2 M) = 0.$$

Here we denote by $E(\tau)$ the matrix $\begin{pmatrix} 0 & \tau & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \tau \\ 0 & \cdots & 0 & 0 \end{pmatrix}$.

Remark 4.4.13. *For the differential equation satisfied by M and the form of the matrix E, one can refer to [15, Section 3.2] for more details.*

Here, M has the form of $\begin{pmatrix} 1 & i\tau s & \cdots & \frac{(i\tau s)^{n_2-1}}{(n_2-1)!} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & i\tau s \\ 0 & \cdots & 0 & 1 \end{pmatrix}$, where τ is a nonzero constant along the generalized bicharacteristic.

Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_{n_2} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ be the canonical basis for $\mathbb{R}^{n_2}$. For any point $\rho_0 \in \operatorname{supp}(\mu_2)$, by the geometric control condition (GCC), we know that there exists a unique general bicharacteristic $s \mapsto \gamma(s)$ such that $\gamma(0) = \rho_0$. Moreover, there exists $\epsilon > 0$, sufficiently small, such that $\gamma((-2\epsilon, 2\epsilon)) \subset \pi^{-1}((0,T) \times \omega)$. Since μ_2 is invariant along the flow associated with M, *i.e.* $\frac{d}{ds}(M^*\mu_2 M) = 0$, we obtain that for any $t_0 \in (0, 2\varepsilon)$, we have

$$\mu_2(0) = M(t_0)^* \mu_2(t_0) M(t_0).$$

Noticing that $\operatorname{supp}(\mu_2)(n_2, n_2) \cap \pi^{-1}((0,T) \times \omega) = \emptyset$ (which also implies that $\mu_2(t_0)e_{n_2} = 0$ by an already developed argument), we obtain that

$$M(-t_0)^* \mu_2(0) M(-t_0) e_{n_2} = \mu_2(t_0) e_{n_2} = 0.$$

Hence, $\mu_2(0)M(-t_0)e_{n_2} = 0$. Moreover, considering $n-1$ times $t_1, \dots t_{n-1}$ such that $t_0 < t_1 < \ldots < t_{n-1} < \varepsilon$, the same argument leads to

$$\begin{cases} \mu_2(0)M(-t_0)e_{n_2} & = 0, \\ \mu_2(0)M(-t_1)e_{n_2} & = 0, \\ \mu_2(0)M(-t_2)e_{n_2} & = 0, \\ \vdots & \\ \mu_2(0)M(-t_{n-1})e_{n_2} & = 0. \end{cases} \tag{4.4.38}$$

From the expression of M, we obtain that $\{M(-t_i)e_{n_2}\}_{i \in [|0, n-1|]}$ is a basis of $\mathbb{R}^n$ (its determinant is proportional to the Vandermonde determinant $\prod_{i<j}(-t_i + t_j)$). Hence, (4.4.38) implies that $\mu_2(0) = 0$. According to the arbitrary choice of $\rho_0 \in \operatorname{supp}(\mu_2)$, we are able to conclude that $\operatorname{supp}(\mu_2) = \emptyset$, *i.e.* $\mu_2 \equiv 0$. Then, we conclude that the relaxed observability inequality (4.4.29) holds for all the solutions of System (4.4.28). □

Step 2: analysis on the invisible solutions

We first define for any $T > 0$ the set of invisible solutions from $]0, T[\times\omega$

$$\begin{aligned}\mathscr{N}_{n_2}(T) = \{\mathscr{W} = (w_1^{1,0}, w_1^{2,0}, \cdots, w_{n_2}^{2,0}, w_1^{1,1}, w_1^{2,1}, \cdots, w_{n_2}^{2,1})^t \in \mathscr{L}_0^{n_2+1} \times \mathscr{L}_{-1}^{n_2+1} \\ \text{such that the associated solution of System (4.4.28)} \\ \text{satisfies } \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} w_1^1(x,t) + w_{n_2}^2(x,t) = 0, \forall (x,t) \in (0,T)\times\omega\}.\end{aligned}$$

With the relaxed observability inequality of (4.4.29), we only need to prove the following key lemma:

Lemma 4.4.14. $\mathscr{N}_{n_2}(T) = \{0\}$.

Proof of Lemma 4.4.14. According to the relaxed observability inequality (4.4.29), for $\mathscr{W} \in \mathscr{N}_{n_2}(T)$, we obtain that

$$||W(0)||^2_{\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}} \leq C||W(0)||^2_{\mathscr{L}_{-1}^{n_2+1}\times\mathscr{L}_{-2}^{n_2+1}}. \tag{4.4.39}$$

We know that $\mathcal{N}_{n_2}(T)$ is a closed subspace of $\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}$. By the compact embedding $L^2(\Omega)\times H^{-1}(\Omega) \hookrightarrow H^{-1}(\Omega)\times H^{-2}(\Omega)$, we know that $\mathcal{N}_{n_2}(T)$ has a finite dimension. Then, we define the operator $\mathscr{A}_{n_2}$ to be the generator associated with System (4.4.28). We know that the solution $(w_1^1, w_1^2, \cdots, w_{n_2}^2, D_t w_1^1, D_t w_1^2, \cdots, D_t w_2^{n_2})^t$ can be written as

$$\begin{pmatrix} w_1^1 \\ w_1^2 \\ \vdots \\ w_{n_2}^2 \\ D_t w_1^1 \\ D_t w_1^2 \\ \vdots \\ D_t w_{n_2}^2 \end{pmatrix} = e^{-t\mathscr{A}_{n_2}}\mathscr{W}.$$

It suffices to prove a unique continuation property for eigenfunctions of the operator $\mathscr{A}_{n_2}$. Let us take $\Phi = (\Phi^0, \Phi^1) = (\phi_1^0, \cdots, \phi_{n_2+1}^0, \phi_1^1, \cdots, \phi_{n_2+1}^1) \in \mathscr{N}_{n_2}(T)$, satisfying

$$\left\{\begin{array}{ll} \mathscr{A}_{n_2}\Phi = & \lambda\Phi, \\ \frac{d_1^2}{(d_1-d_2)^2}\phi_1^0 + \phi_{n_2+1}^0 & = 0 \text{ in } \omega. \end{array}\right. \tag{4.4.40}$$

Then, it is equivalent to a the system

$$\left\{\begin{array}{ll} (-D\Delta_D + A^*)\varphi & = \lambda^2\varphi, \\ \hat{b}^*\varphi|_\omega & = 0. \end{array}\right. \tag{4.4.41}$$

Indeed, as explained in Remark 4.3.13, Φ and φ verify the relation $\varphi = \mathcal{S}'(\lambda, \Delta)\Phi$ (where we replace formally D_t by λ). The study of (4.4.41) is totally similar to the one of (4.3.57): using the analyticity, we know that $\hat{b}^*\varphi \equiv 0$. Then, we obtain that $\hat{b}^*(-D\Delta_D + A^*)^k\varphi = 0$, for any $k \in \mathbb{N}$, *i.e.* $\varphi \in Ker(\mathcal{K}^*) = \{0\}$, so that $\varphi \equiv 0$, which concludes our proof. $\square$

4.4.2 Reformulation of the system in the general case

According to Proposition 4.1.8, we already know that the operator Kalman rank condition is necessary for the exact controllability of System (4.1.1). In this section, provided with the operator Kalman rank condition $Ker(\mathcal{K}^*) = \{0\}$, we plan to give a reformulation of System (4.1.1).

As a consequence of Proposition 4.1.6, we know that (A_2, B) satisfies Kalman rank condition. Therefore, applying Theorem 4.3.1, there exists an invertible matrix P such that we reformulate System (4.1.1) into the following system

$$\begin{cases} \Box_1 \tilde{u}_1^1 + \sum_{j=1}^{n_2} \tilde{\alpha}_j \tilde{u}_j^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 \tilde{u}_1^2 + \tilde{u}_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box_2 \tilde{u}_{n_2-1}^2 + \tilde{u}_{n_2}^2 = 0 \text{ in } (0,T)\times\Omega, & & \\ \Box_2 \tilde{u}_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j} \tilde{u}_j^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ \tilde{u}_1^1 = 0, \tilde{u}_1^2 = \cdots \tilde{u}_{n_2}^2 & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\tilde{u}_1^1, \tilde{u}_1^2, \cdots, \tilde{u}_{n_2}^2)|_{t=0} & = (\tilde{u}_1^{1,0}, \tilde{u}_1^{2,0}, \cdots, \tilde{u}_{n_2}^{2,0}) & \text{in } \Omega, \\ (\partial_t\tilde{u}_1^1, \partial_t\tilde{u}_1^2, \cdots, \partial_t\tilde{u}_{n_2}^2)|_{t=0} & = (\tilde{u}_1^{1,1}, \tilde{u}_1^{2,1}, \cdots, \tilde{u}_{n_2}^{2,1}) & \text{in } \Omega, \end{cases} \tag{4.4.42}$$

where $\tilde{u}_1^1 = u_1^1$, $\tilde{U}_2 = PU_2$ and $(\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n) = (\alpha_1, \cdots, \alpha_n)P^{-1}$. Define $s = \max\{1 \leqslant j \leqslant n_2; \tilde{\alpha}_j \neq 0\}$. From Proposition 4.4.1, the appropriate state space for (4.4.42) is $\mathcal{H}_1^s \times \mathcal{H}_0^s$. Moreover, by Theorem 4.4.8, under our hypotheses, we have exact controllability of System (4.4.42) in the state space $\mathcal{H}_1^s \times \mathcal{H}_0^s$. This immediately leads to the conclusion of Theorem 4.1.16.

4.5 Some comments

As we can see, the system (4.1.2) is only an example of a more general system as follows:

$$\begin{cases} (\partial_t^2 - D\Delta_D)U + AU & = \hat{b}f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) & \text{in } (0,T)\times\Omega, \\ U & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (U, \partial_t U)|_{t=0} & = (U^0, U^1) & \text{in } \Omega, \end{cases} \tag{4.5.1}$$

with here

$$D = \begin{pmatrix} d_1 Id_{n_1} & 0 \\ 0 & d_2 Id_{n_2} \end{pmatrix}_{n\times n}, A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{n\times n},$$
$$\hat{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_{n\times m}, \text{ and } f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}_{m\times 1} \tag{4.5.2}$$

where $n = n_1 + n_2$ and $f_j \in L^2((0,T)\times\omega), j = 1, 2, \cdots, m$. In this very general system (4.5.1), there are three different kinds of effective parts acting on the controllability problem, that is, control functions and two different types of coupling.

The first part is obviously the control functions. The more control functions we have, the more sophisticated structure we demand for the coupled matrix to obtain the controllability. It is very related to the Brunovský Normal Form and when we consider more than one control function, the standard Brunovský Normal Form has more than one block in the coupling matrix, which increases the complicity of the calculation to obtain an explicit formula of the compatibility conditions (as we have seen, for instance, in (4.1.8)). However, when we deal with the case with more than one control functions, we usually rely on the Brunovský Normal Form to put the coupling matrix into the standard form and then, deal with the problem block by block. This means that we first need to establish the result with only one block, *i.e.* with only one control function. In the system (4.1.2), we choose that $\tilde{b}$ only acts on the second part of the system. The reason is that if we give both parts the effective control function, we cannot observe the influence of the coupling term because of the regularity.

The second part we considered is the coupling with the same speed, which corresponds to A_{11} and A_{22}, and on the other hand, the third part is the coupling effects of the different speeds, which corresponds to A_{12} and A_{21}. As we can see in the proof of the Theorem 4.1.16, coupling with same speed, we are able to observe a phenomena of regularity increase by one with successive solutions. While we can prove that the regularity gap between two coupled solutions with different speeds is two (one can see in Subsection 4.2.2). This difference gives us the motivation to consider that the simplest example of coupled wave system containing the two different coupling effects, *i.e.* the system (4.1.2). We try to use this example to analyse the different influence of these two types of coupling terms. When one introduces the fully coupling matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{n\times n}$, it is complicated to analyse the two different types of coupling. Because they are combined too closely, it is difficult to separate them. From a technical point of view, it seems very hard to derive an appropriate normal form similar to Brunovský form to obtain the compatibility conditions and the appropriate state space.

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4.6 Appendix I: On the operator Kalman rank condition

Proof of Proposition 4.1.6. Let $\lambda \in \sigma(-\Delta_D)$ and $\mathcal{K}(\lambda) = [(\lambda D+A)|\hat{b}] \in \mathcal{M}_n(\mathbb{R})$ (remind that $\hat{b} = {}^t(0,b) \in \mathbb{R}^n$). Firstly, we compute the form of the matrix $\mathcal{K}(\lambda)$ by induction.

$$\mathcal{K}(\lambda) = \begin{pmatrix} S_{n-1}(\lambda) & \cdots & S_j(\lambda) & \cdots & A_1 b & 0 \\ (d_2\lambda + A_2)^{n-1} b & \cdots & (d_2\lambda + A_2)^j b & \cdots & (d_2\lambda + A_2) b & b \end{pmatrix}. \tag{4.6.1}$$

The general term $S_j(\lambda)$, $1 \leq j \leq n-1$ is defined by

$$S_j(\lambda) = A_1 \left(\sum_{k=0}^{j-1} d_1^k \lambda^k (d_2\lambda + A_2)^{j-1-k} \right) b. \tag{4.6.2}$$

Since the rank of a matrix is invariant under elementary operations on the columns (that we will shorten in column transformation in what follows), it is easy to see that $rank(\mathcal{K}(\lambda)) = rank(\tilde{\mathcal{K}}(\lambda))$, where

$$\tilde{\mathcal{K}}(\lambda) = \begin{pmatrix} \tilde{S}_{n-1}(\lambda) & \cdots & \tilde{S}_j(\lambda) & \cdots & A_1 b & 0 \\ A_2^{n-1} b & \cdots & A_2^j b & \cdots & A_2 b & b \end{pmatrix}, \tag{4.6.3}$$

with

$$\tilde{S}_j(\lambda) = A_1 \left(\sum_{k=0}^{j-1} (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right) b. \tag{4.6.4}$$

Let us first prove the necessity of the conditions. Suppose that $n_1 > 1$ and let us prove that the Kalman matrix $K(\lambda)$ is not of full rank. We take the n_1-th column of the matrix $\tilde{\mathcal{K}}(\lambda)$, *i.e.*

$$\begin{pmatrix} \tilde{S}_{n_2}(\lambda) \\ A_2^{n_2} b \end{pmatrix}.$$

Let $\chi(X) = X^{n_2} + \sum_{j=0}^{n_2-1} a_j X^j$ be the characteristic polynomial of the matrix A_2. By the Cayley-Hamilton Theorem, $A_2^{n_2} = -\sum_{j=0}^{n_2-1} a_j A_2^j$.

By using an adequate column transformation, we can put the n_1-th column into the form

$$\begin{pmatrix} T_{n_2}(\lambda) \\ 0 \end{pmatrix}, \tag{4.6.5}$$

where $T_{n_2}(\lambda) = \tilde{S}_{n_2}(\lambda) + \sum_{j=1}^{n_2-1} a_j \tilde{S}_j(\lambda)$. By (4.6.4),

$$\begin{aligned} \sum_{j=1}^{n_2-1} a_j \tilde{S}_j(\lambda) &= \sum_{j=1}^{n_2-1} a_j A_1 \left(\sum_{k=0}^{j-1} (d_1-d_2)^k \lambda^k A_2^{j-1-k} \right) b \\ &= A_1 \left(\sum_{k=0}^{n_2-2} \sum_{j=k+1}^{n_2-1} a_j (d_1-d_2)^k \lambda^k A_2^{j-1-k} \right) b. \end{aligned}$$

Using the expression of $\tilde{S}_{n_2}(\lambda)$ given in (4.6.4), we obtain that

$$\begin{aligned} T_{n_2}(\lambda) &= \tilde{S}_{n_2}(\lambda) + \sum_{j=1}^{n_2-1} a_j \tilde{S}_j(\lambda) \\ &= A_1 \left(\sum_{k=0}^{n_2-1} (d_1-d_2)^k \lambda^k A_2^{j-1-k} \right) b + A_1 \left(\sum_{k=0}^{n_2-2} \sum_{j=k+1}^{n_2-1} a_j (d_1-d_2)^k \lambda^k A_2^{j-1-k} \right) b \\ &= A_1 \left(\sum_{k=0}^{n_2-2} (d_1-d_2)^k \lambda^k \left(A_2^{n_2-1-k} + \sum_{j=k+1}^{n_2-1} a_j A_2^{j-1-k} \right) + (d_1-d_2)^{n_2-1} \lambda^{n_2-1} \right) b, \end{aligned}$$

i.e.

$$T_{n_2}(\lambda) = A_1 \left(\sum_{k=0}^{n_2-2} (d_1-d_2)^k \lambda^k \sum_{j=k+1}^{n_2} a_j A_2^{j-1-k} + (d_1-d_2)^{n_2-1} \lambda^{n_2-1} \right) b. \tag{4.6.6}$$

Here and hereafter, we use the notation $a_{n_2} = 1$ in order to obtain a clean from. Now, we take the $(n_1-1)-$th column of the matrix $\tilde{\mathcal{K}}(\lambda)$, *i.e.*

$$\begin{pmatrix} \tilde{S}_{n_2+1}(\lambda) \\ A_2^{n_2+1} b \end{pmatrix}$$

Again using the characteristic polynomial of the matrix A_2, we obtain that

$$\begin{aligned}
A_2^{n_2+1} &= -A_2 \sum_{j=0}^{n_2-1} a_j A_2^j \\
&= -\sum_{j=0}^{n_2-2} a_j A_2^{j+1} - a_{n_2-1} A_2^{n_2} \\
&= -\sum_{j=0}^{n_2-2} a_j A_2^{j+1} + a_{n_2-1} \sum_{j=0}^{n_2-1} a_j A_2^j \\
&= \sum_{j=1}^{n_2-1} (a_j a_{n_2-1} - a_{j-1}) A_2^j + a_{n_2-1} a_0.
\end{aligned}$$

By applying an adequate column transformation, we can put the (n_1-1)-th column into the form:

$$\begin{pmatrix} T_{n_2+1}(\lambda) \\ 0 \end{pmatrix},$$

where $T_{n_2+1}(\lambda)$ satisfies

$$\begin{aligned}
T_{n_2+1}(\lambda) &= \tilde{S}_{n_2+1}(\lambda) - \sum_{j=1}^{n_2-1} (a_j a_{n_2-1} - a_{j-1}) \tilde{S}_j(\lambda) \\
&= A_1 \left(\sum_{k=0}^{n_2} (d_1 - d_2)^k \lambda^k A_2^{n_2-k} \right) b \\
&- \sum_{j=1}^{n_2-1} (a_j a_{n_2-1} - a_{j-1}) A_1 \left(\sum_{k=0}^{j-1} (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right) b \\
&= A_1 \left(\sum_{k=0}^{n_2} (d_1 - d_2)^k \lambda^k A_2^{n_2-k} \right) b \\
&- A_1 \left(\sum_{k=0}^{n_2-2} \sum_{j=k+1}^{n_2-1} (a_j a_{n_2-1} - a_{j-1}) (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right) b \\
&= A_1 \left(\sum_{k=0}^{n_2-2} (d_1 - d_2)^k \lambda^k A_2^{n_2-k} + (d_1 - d_2)^{n_2} \lambda^{n_2} + (d_1 - d_2)^{n_2-1} \lambda^{n_2-1} A_2 \right) b \\
&+ A_1 \left(\sum_{k=0}^{n_2-2} \sum_{j=k+1}^{n_2-1} a_{j-1} (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right. \\
&\left. - a_{n_2-1} \sum_{k=0}^{n_2-2} \sum_{j=k+1}^{n_2-1} a_j (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right) b.
\end{aligned}$$

Now consider the sum

$$\sum_{k=0}^{n_2-2}(d_1-d_2)^k\lambda^k A_2^{n_2-k}+\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2-1}a_{j-1}(d_1-d_2)^k\lambda^k A_2^{j-1-k}$$
$$=\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(A_2^{n_2-k}+\sum_{j=k+1}^{n_2-1}a_{j-1}A_2^{j-1-k}\right)+A_2^{n_2}+\sum_{j=1}^{n_2-1}a_{j-1}A_2^{j-1}$$
$$=\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}\right)-\sum_{j=0}^{n_2-1}a_jA_2^j$$
$$+\sum_{j=1}^{n_2-1}a_{j-1}A_2^{j-1}-\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k a_{n_2-1}A_2^{n_2-1-k}$$

$$=\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}\right)-\sum_{j=0}^{n_2-1}a_jA_2^j$$
$$+\sum_{j=1}^{n_2-1}a_{j-1}A_2^{j-1}-\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k a_{n_2-1}A_2^{n_2-1-k}$$
$$=\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}\right)-a_{n_2-1}A_2^{n_2-1}$$
$$-a_{n_2-1}\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k A_2^{n_2-1-k}$$
$$=\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}\right)-a_{n_2-1}\sum_{k=0}^{n_2-2}(d_1-d_2)^k\lambda^k A_2^{n_2-1-k}.$$

Therefore, we obtain

$$\begin{aligned}
T_{n_2+1}(\lambda) &= A_1\left(\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}\right)b\\
&+A_1\left(-a_{n_2-1}\sum_{k=0}^{n_2-2}(d_1-d_2)^k\lambda^kA_2^{n_2-1-k}+(d_1-d_2)^{n_2}\lambda^{n_2}\right.\\
&\left.+(d_1-d_2)^{n_2-1}\lambda^{n_2-1}A_2\right)b+A_1\left(-a_{n_2-1}\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2-1}a_j(d_1-d_2)^k\lambda^kA_2^{j-1-k}\right)b\\
&=A_1\left(\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}+(d_1-d_2)^{n_2}\lambda^{n_2}\right)b\\
&+A_1\left((d_1-d_2)^{n_2-1}\lambda^{n_2-1}A_2-a_{n_2-1}\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2}a_j(d_1-d_2)^k\lambda^kA_2^{j-1-k}\right)b.
\end{aligned}$$

Then, we aim to find a connection between the terms $T_{n_2+1}(\lambda)$ and $T_{n_2}(\lambda)$. By calculation, we obtain

$$\begin{aligned}
(d_1-d_2)\lambda T_{n_2}(\lambda) &= A_1\left(\sum_{k=0}^{n_2-2}(d_1-d_2)^{k+1}\lambda^{k+1}\sum_{j=k+1}^{n_2}a_jA_2^{j-1-k}+(d_1-d_2)^{n_2}\lambda^{n_2}\right)B\\
&=A_1\left(\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\sum_{j=k}^{n_2}a_jA_2^{j-k}+(d_1-d_2)^{n_2}\lambda^{n_2}\right)B\\
&+(d_1-d_2)^{n_2-1}\lambda^{n_2-1}A_1A_2B\\
&=T_{n_2+1}(\lambda)+A_1\left(a_{n_2-1}\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2}a_j(d_1-d_2)^k\lambda^kA_2^{j-1-k}\right)B\\
&=T_{n_2+1}(\lambda)+a_{n_2-1}T_{n_2}(\lambda).
\end{aligned}$$

Hence, we know that $T_{n_2+1}(\lambda)=((d_1-d_2)\lambda-a_{n_2-1})\,T_{n_2}(\lambda)$, which means that the two columns

$$\begin{pmatrix}T_{n_2}(\lambda)\\0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}T_{n_2+1}(\lambda)\\0\end{pmatrix}$$

are linearly dependent. This means that

$$\begin{pmatrix}\tilde{S}_{n_2}(\lambda)\\A_2^{n_2}b\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\tilde{S}_{n_2+1}(\lambda)\\A_2^{n_2+1}b\end{pmatrix}$$

are linearly dependent. By the expression of $\tilde{\mathcal{K}}(\lambda)$ given in (4.6.3) and the definition of $\tilde{S}_j$ given in (4.6.4), we deduce that all the j-th columns of $\tilde{\mathcal{K}}(\lambda)$, for $j\leqslant n_1$, are linearly dependent. We deduce that $\tilde{K}(\lambda)$ is of rank less that $n - n_1 + 1 = n_2 + 1$. This is in contradiction with the fact that $\tilde{\mathcal{K}}(\lambda) \in \mathcal{M}_n(\mathbb{R})$ is of full rank $n = n_1 + n_2 > n_2 + 1$ since we assumed that $n_1 > 1$. So we deduce that $n_1 = 1$.

Concerning the two other conditions, remark that the first column of $\tilde{K}(\lambda)$ can be changed by a previously introduced column transformation into (4.6.5), where $T_{n_2}(\lambda)$ verifies (4.6.6). We deduce that the rank of $K(\lambda)$ is equal to the rank of the matrix

$$\begin{pmatrix} T_{n_2}(\lambda) & \tilde{S}_{n_2-1}(\lambda) & \cdots & \tilde{S}_j(\lambda) & \cdots & A_1 b & 0 \\ 0 & A_2^{n_2-1} & \cdots & A_2^{j-1} b & \cdots & A_2 b & b \end{pmatrix}.$$

This matrix is of full rank $n = n_2 + 1$ (if and) only if $T_{n_2}(\lambda) \neq 0$ (which gives (4.1.4) thanks to (4.6.6)) and

$$\begin{pmatrix} A_2^{n_2-1} & \cdots & A_2^{j-1} b & \cdots & A_2 b & b \end{pmatrix} \in \mathcal{M}_{n_2,n_2}(\mathbb{R})$$

is of full rank n_2, which is exactly meaning that (A_2, b) verifies the usual Kalman rank condition.

The sufficiency of the three conditions given in Proposition 4.1.6 is also straightforward, by the same arguments.

■

4.7 Appendix II: Proof of Lemma 4.4.7

We first look at $u^2_{n_2}$. Since $j+k \leq n_2$, we know for $j = n_2$, the conclusion is trivial. For $1 \leq j \leq n_2 - 1$, we argue by induction. When $k = 0$, the conclusion holds for sure. Assume that

$$D_t^{2k-2} u_j^2 = \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2 \Delta)^l u^2_{j+k-1-l}. \tag{4.7.1}$$

Then for $D_t^{2k} u_j^2$, we know that

$$\begin{aligned} D_t^{2k} u_j^2 &= D_t^2 D_t^{2k-2} u_j^2 \\ &= \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2 \Delta)^l D_t^2 u^2_{j+k-1-l}. \end{aligned} \tag{4.7.2}$$

Using the equation $D_t^2 u_{j+k-1-l}^2 = -d_2 \Delta u_{j+k-1-l}^2 + u_{j+k-l}^2$, we obtain that

$$\begin{aligned} D_t^{2k} u_j^2 &= \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2\Delta)^{l+1} u_{j+k-1-l}^2 + \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2\Delta)^l u_{j+k-l}^2 \\ &= \sum_{l=1}^{k} \binom{k-1}{l-1} (-d_2\Delta)^l u_{j+k-l}^2 + \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2\Delta)^l u_{j+k-l}^2 \\ &= \sum_{l=1}^{k-1} (\binom{k-1}{l-1} + \binom{k-1}{l}) (-d_2\Delta)^l u_{j+k-l}^2 + (-d_2\Delta)^k u_j^2 + u_{j+k}^2. \end{aligned} \tag{4.7.3}$$

Since $(\binom{k-1}{l-1} + \binom{k-1}{l}) = \binom{k}{l}$, we obtain the conclusion.

■

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ÉCOLE DOCTORALE
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Titre : Contrôlabilité de systèmes des ondes couplées

Mots clés : contrôlabilité, systèmes couplés, équation d'onde

Résumé : Dans cette thèse, nous étudions les théories étroitement liées du contrôle et les propriétés de la continuation unique, pour des équations et systèmes des ondes linéaires.
Nous avons étudié la contrôlabilité simultanée des systèmes des ondes dans un domaine ouvert de $\mathbb{R}^d$. Nous obtenons un résultat de contrôlabilité partielle sur un espace co-dimensionnel fini pour des équations d'onde couplées par une seule fonction de contrôle. Pour la propriété de continuation unique des fonctions propres, nous avons donné un contre-exemple pour montrer que dans certaines métriques, la propriété de continuation unique n'est pas vraie. De plus, nous avons étudié différentes conditions pour garantir la propriété de continuation unique. Nous avons étudié également notre résultat au cas de coefficients constants et éventuellement de fonctions de contrôle multiples. Dans ce contexte, nous avons prouvé que la propriété de contrôlabilité est équivalente à une condition de rang de Kalman appropriée.
Nous avons étudié un problème de contrôlabilité exact dans un domaine ouvert Ω de $\mathbb{R}^d$, pour un système des ondes couplées, avec des vitesses différentes et une seule commande agissant sur une sous-ensemble ouvert ω satisfaisant la condition de contrôle géométrique et sur une seule vitesse. Les actions pour les équations des ondes avec la deuxième vitesse sont obtenues par un terme decouplage. Tout d'abord, nous construisons des espaces d'états appropriés avec desconditions de compatibilité associées à la structure de couplage. Deuxièmement, dans ces espaces bien préparés, nous prouvons que le système des ondes couplées est exactement contrôlable si et seulement si la structure de couplage satisfait à une condition de rang de Kalman de l'opérateur.

Title: The Controllability of the Coupled Wave Systems

Keywords: Controllability, Coupled systems, Wave equations

Abstract: We study the simultaneous controllability of wave systems in an open domain of $\mathbb{R}^d$. We obtain a partial controllability result on a finite co-dimensional space for wave equations coupled by a single control function. For the unique continuation property of eigenfunctions, we construct a counterexample to show that in some metrics, the unique continuation property does not hold. Moreover, we study different conditions to ensure the unique continuation property. We also extend our result to the case of constant coefficients and possibly multiple control functions. In this context, we prove the controllability property is equivalent to an appropriate Kalman rank condition.
We also consider an exact controllability problem in a smooth bounded domain Ω of $\mathbb{R}^d$, for a coupled wave system, with different speeds and a single control acting on an open subset ω satisfying the Geometric Control Condition and on one speed only. Actions for the wave equations with the second speed are obtained through a coupling term. Firstly, we construct appropriate state spaces with compatibility conditions associated with the coupling structure. Secondly, in these well-prepared spaces, we prove that the coupled wave system is exactly controllable if and only if the coupling structure satisfies an operator Kalman rank condition.