The controllability of a special class of coupled wave system

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Preliminary-controllability

Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary. Consider $\omega \subset \Omega$ to be a subdomain. This is the basic geometric setting for the interior control problem.

$$(\partial_t^2 - \Delta)u = f\mathbb{1}_\omega(x)\mathbb{1}_{(0,T)}(t), u|_{\partial\Omega} = 0,$$

where $f \in L^2((0,T) \times \omega)$, For this model, we say the wave equation is controllable in time $T > 0$ if:

Exact controllability

For any initial data $(u_0, u_1) \in H_0^1 \times L^2$ and any target $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$, there exists $f \in L^2((0,T) \times \omega)$ such that the solution u satisfies $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ and $(u, \partial_t u)|_{t=T} = (\tilde{u}_0, \tilde{u}_1)$.

Kalman conditions

Definition (Usual algebraic Kalman rank condition)

Let m, n be two positive integers. Assume $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_{n,m}(\mathbb{R})$. We introduce the Kalman matrix associated to A and B given by $[A|B] = [A^{n-1}B|\cdots|AB|B] \in \mathcal{M}_{n,nm}(\mathbb{R})$. We say that (A, B) satisfies the Kalman rank condition if $[A|B]$ is of full rank.

Definition (Kalman operator)

Assume that $X \in \mathcal{M}_n(\mathbb{R})$ and $Y \in \mathcal{M}_{n,m}(\mathbb{R})$. Moreover, let D be a diagonal matrix. Then, the Kalman operator associated with $(-D\Delta + X, Y)$ is the matrix operator $\mathscr{K} = [-D\Delta + X|Y] : D(\mathscr{K}) \subset (L^2)^{nm} \to (L^2)^n)$.

Definition (Operator Kalman rank condition)

We say that the Kalman operator $\mathscr{K}$ satisfies the operator Kalman rank condition if $Ker(\mathscr{K}^*) = \{0\}$.

Geometric Control Condition

Let p_g be the principal symbol of the operator $\partial_t^2 - \Delta_g$.

Definition

For $\omega \subset \Omega$ and $T > 0$, we shall say that the pair (ω, T, p_g) satisfies GCC if every general bicharacteristic of p_g meets ω in a time $t < T$.

This is a very important condition when one considers the control of waves. One can refer Rauch-Taylor 74', Bardos-Lebeau-Rauch 88',92', Burq-Gérard 97',$\cdots$

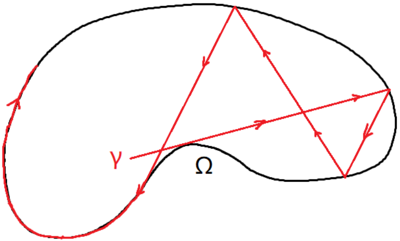

Figure: General bicharacteristics

Microlocal defect measure-1

Based on Gérard-Leichtnam 93' and Burq 97'. Let $(u^k)_{k\in\mathbb{N}}$ be a bounded sequence in $\left(L^2_{loc}(\mathbb{R}^+; L^2(\Omega))\right)^n$, converging weakly to 0 and such that

$$\begin{cases} Pu^k = o(1)_{H^{-1}}, \\ u^k|_{\partial M} = 0. \end{cases} \tag{1}$$

Let $\underline{u}^k$ be the extension by 0 across the boundary of Ω. Then the sequence $\underline{u}^k$ is bounded in $\left(L^2_{loc}(\mathbb{R}_t; L^2(\mathbb{R}^d))\right)^n$. Let $\underline{\mathcal{A}}$ be the space of $n\times n$ matrices of classical pseudo-differential operators of order 0 with compact support in $\mathbb{R}^+\times\mathbb{R}^d$

Proposition

There exists a subsequence of $(\underline{u}^k)$ (still noted by $(\underline{u}^k)$) and $\underline{\mu}\in\underline{\mathcal{M}}^+$ such that

$$\forall A\in\underline{\mathcal{A}},\quad \lim_{k\to\infty}(A\underline{u}^k,\underline{u}^k)_{L^2} = \langle\underline{\mu},\sigma(A)\rangle, \tag{2}$$

where $\sigma(A)$ is the principal symbol of the operator A (which is a matrix of smooth functions, homogeneous of order 0 in the variable ξ, i.e. a function on $S^*((\mathbb{R}^+\times\mathbb{R}^d))$.

Microlocal defect measure-1

For the microlocal defect measure $\underline{\mu}$ defined before, we have the following properties.

- $\text{supp}(\underline{\mu}) \subset Char(P) = \{(t, x, \tau, \xi); x \in \overline{M}, \tau^2 = |\xi|_x^2\}$.
- The measure $\underline{\mu}$ does not charge the hyperbolic points in ∂M :

$$\underline{\mu}(\mathcal{H}) = 0.$$

- $\underline{\mu}$ is invariant along the generalized bicharacteristic flow.

Classic Results

Theorem

If (ω, T, p) satisfies the GCC, then the equation:

$$(\partial_t^2 - \Delta)u = f\mathbb{1}_\omega(x)\mathbb{1}_{(0,T)}(t)$$

is exact controllable in $H_0^1(\Omega) \times L^2(\Omega)$.

General approach

- Apply Hilbert uniqueness method and get the adjoint system and observability inequality;
- Deal with high frequency part using the property of the defect measure;
- Deal with the low frequency part using the uniqueness facts.

Observability

Applying HUM, we obtain the observability inequality

$$\int_0^T \int_\omega |v|^2 dxdt \geq C \left(||v^0||_{L^2}^2 + ||v^1||_{H^{-1}}^2 \right)$$

for the adjoint equation $(\partial_t^2 - \Delta)v = 0$, with initial data (v^0, v^1).

1. High frequency: $||v(0)||^2_{L^2 \times H^{-1}} \leq C(\int_0^T \int_\omega |v|^2 dxdt + ||v(0)||^2_{H^{-1} \times H^{-2}})$
2. Low frequency: unique continuation result.

High Frequency Estimates

To establish the weak observability, we prove by contradiction argument. Assume the weak observability is false, there exists a sequence $(v^k(0))_{k\in\mathbb{N}}$ such that

$$||v^k(0)||^2_{L^2\times H^{-1}} = 1$$

$$\int_0^T \int_\omega |v^k|^2 dxdt \to 0$$

$$||v^k(0)||^2_{H^{-1}\times H^{-2}} \to 0$$

$\Rightarrow \exists\mu$: vanishes in $(0, T) \times \omega$ and invariant along the bicharacteristics.
$\Rightarrow \mu \equiv 0$ (using GCC).

Low Frequency Estimates

Using contradiction argument, to obtain the observability is reduced to a unique continuation problem:

$$\left.\begin{aligned} -\Delta v &= \beta^2 v, \\ v|_\omega &= 0, \\ v &\in H_0^1(\Omega). \end{aligned}\right\} \Rightarrow v \equiv 0.$$

In fact, we know $v = 0$ by the Carleman estimates for elliptic operator $-\Delta$.

Same control for different speeds: a simple model

$$\begin{cases} (\partial_t^2 - \Delta)u_1 = f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \\ (\partial_t^2 - 2\Delta)u_2 = f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) \\ u_j = 0 \quad \text{on } (0,T)\times\partial\Omega, j = 1,2, \\ u_j(0,x) = u_j^0(x) \in H_0^1, \quad \partial_t u_j(0,x) = u_j^1(x) \in L^2, j = 1,2. \end{cases} \tag{M1}$$

Question

Is this system controllable in $(H_0^1 \times L^2)^2$?

Theorem

Assume that (ω, T, p_i), $i = 1,2$ satisfies the GCC, then the system (M1) is exactly controllable.

Brief proof

Observability for (M1): $||V(0)||^2_{(L^2\times H^{-1})^2} \leq C\int_0^T\int_\omega |v_1+v_2|^2 dxdt$, where (v_1, v_2) solves

$$\begin{cases}(\partial_t^2-\Delta)v_1=0,\\ (\partial_t^2-2\Delta)v_2=0,\end{cases}$$

with initial data $V(0)$.

- High frequency: $||v_1+v_2||^2_{L^2} = ||v_1||^2_{L^2} + ||v_2||^2_{L^2} + o(1)$,
- Low frequency: uniqueness result:

$$\left.\begin{aligned}-\Delta e_1 &= \lambda^2 e_1,\\ -2\Delta e_2 &= \lambda^2 e_2,\\ (e_1+e_2)|_\omega &= 0.\end{aligned}\right\} \Rightarrow e_1=e_2=0.$$

Comments

In general, we are able to deal with simultaneous control problem for n wave equations with n different metric $g_1, g_2, \cdots, g_n$ if we assume:

- $g_1 < g_2 < \cdots < g_n$ (generalization of different speeds),
- ω satisfies GCC for $g_1, g_2, \cdots, g_n$,
- Unique continuation properties.

Motivations: a simple model

$$\begin{cases} (\partial_t^2 - \Delta)u_1 + u_2 & = 0 \quad \text{in } (0,T)\times\Omega, \\ (\partial_t^2 - 2\Delta)u_2 + u_3 & = 0 \quad \text{in } (0,T)\times\Omega, \\ (\partial_t^2 - 2\Delta)u_3 & = f\mathbf{1}_\omega \quad \text{in } (0,T)\times\Omega, \end{cases} \tag{M2}$$

with the Dirichlet boundary condition and some initial data. This system has the following features:

- f is only acting directly on u_3,
- u_2 and u_3 are coupled via a weak coupling (lower order),
- u_1 and u_2 are coupled via a very weak coupling (lower order+different speed).

$\Rightarrow$ Compatibility conditions.

Question

- What is the appropriate state space for this system (M2)?
- Is it controllable?

Sobolev spaces with boundary condition

Assume

$$-\Delta e_j = \beta_j^2 e_j, \quad ||e_j||_{L^2} = 1.$$

$H^s_\Omega(\Delta)$ is the Hilbert space defined by

$$H^s_\Omega(\Delta) = \{u = \sum_{j\in\mathbb{N}^*} a_j e_j; \sum_{j\in\mathbb{N}^*} (1+\beta_j^2)^s |a_j|^2 < \infty\}.$$

This is a suitable Sobolev space assocaited with the Dirichlet Boundary conditions.

On regularity of the system (M2)

For this example system, the controllability from zero is equivalent to the null controllability. Therefore, we begin with zero initial conditions.

$(u_1, u_2, u_3) \in H^4_\Omega \times H^2_\Omega \times H^1_\Omega$

In fact, it is classic to prove that

$$u_3 \in C^0([0,T], H^1_\Omega) \cap C^1([0,T], H^0_\Omega),$$
$$u_2 \in C^0([0,T], H^2_\Omega) \cap C^1([0,T], H^1_\Omega).$$

For u_1, $\Box_1 u_1 = -u_2$, which implies that $\Box_2 \Box_1 u_1 = -\Box_2 u_2 = u_3$. Hence, we obtain that $\Box_2 u_1 \in C^0 H^2_\Omega \cap C^1 H^1_\Omega$. And we already know that $\Box_1 u_1 = -u_2 \in C^0 H^2_\Omega \cap C^1 H^1_\Omega$. Take the difference, we obtain that $\Delta u_1 \in C^0 H^2_\Omega \cap C^1 H^1_\Omega$ which implies that $u_1 \in C^0 H^4_\Omega \cap C^1 H^3_\Omega$.

Compatibility conditions

$(-\Delta)^2 u_1 + \Delta u_2 \in H^1_0$.

Compatibility conditions

We introduce a transform $\mathcal{S}$ by

$$\mathcal{S}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} D_t^3 u_1, \\ D_t u_2, \\ u_3. \end{pmatrix}.$$

Moreover, (v_1, v_2, v_3) satisfies the following system:

$$\begin{cases} \Box_1 v_1 + D_t^2 v_2 = 0 \text{ in } (0, T) \times \Omega, \\ \Box_2 v_2 + D_t v_3 = 0 \text{ in } (0, T) \times \Omega, \\ \Box_2 v_3 = f \text{ in } (0, T) \times \Omega. \end{cases} \tag{M2v}$$

Using the identity $-D_t^2 = 2\Box_1 - \Box_2$, we have that $\Box_1(v_1 - 2v_2) - D_t v_3 = 0$. Hence, $\Box_1(D_t v_1 - 2D_t v_2 + 2v_3) = f$. However, we know that $D_t v_1 - 2D_t v_2 + 2v_3 = (-\Delta)^2 u_1 + \Delta u_2 + u_3$, which implies that $(-\Delta)^2 u_1 + \Delta u_2 \in H^1_\Omega$.

A wave system coupled with different speeds

We aim to deal with some controllability properties of the following type of coupled wave systems:

$$\begin{cases} (\partial_t^2 - D\Delta)U + AU & = \hat{b} f \mathbf{1}_\omega & \text{in } (0,T)\times\Omega, \\ U & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (U,\partial_t U)|_{t=0} & = (U^0,U^1) & \text{in } \Omega, \end{cases} \tag{CWS}$$

with here

$$D = \begin{pmatrix} d_1 Id_{n_1} & 0 \\ 0 & d_2 Id_{n_2} \end{pmatrix}_{n\times n}, A = \begin{pmatrix} 0 & A_1 \\ 0 & A_2 \end{pmatrix}_{n\times n}, \text{ and } \hat{b} = \begin{pmatrix} 0 \\ b \end{pmatrix}_{n\times 1},$$

where $n = n_1 + n_2$ and $d_1 \neq d_2$. $A_1 \in \mathcal{M}_{n_1,n_2}(\mathbb{R})$ and $A_2 \in \mathcal{M}_{n_2}(\mathbb{R})$ are two given coupling matrices and $b \in \mathbb{R}^{n_2}$.

Equivalent operator Kalman rank condition

Proposition

We denote by $\mathcal{K} = [-D\Delta + A|\hat{b}]$ *the Kalman operator associated with System* (CWS)*. Then,* $Ker(\mathcal{K}^*) = \{0\}$ *is equivalent to satisfying all the following conditions:*

1. $n_1 = 1$;
2. (A_2, b) *satisfies the usual Kalman rank condition;*
3. *Assume that* $A_1 = \alpha = (\alpha_1, \cdots, \alpha_{n_2})$*. Then* $\forall \lambda \in \sigma(-\Delta)$*,* α *satisfies*

$$\alpha \left(\sum_{k=0}^{n_2-2} (d_1 - d_2)^k \lambda^k \sum_{j=k+1}^{n_2} a_j A_2^{j-1-k} + (d_1 - d_2)^{n_2-1} \lambda^{n_2-1} Id_{n_2} \right) \widehat{b} \neq 0, \tag{KC}$$

where $(a_j)_{0 \le j \le n_2}$ *are the coefficients of the the characteristic polynomial of the matrix* A_2*, i.e.* $\chi(X) = X^{n_2} + \sum_{j=0}^{n_2-1} a_j X^j$*, with the convention that* $a_{n_2} = 1$*.*

Controllability of the coupled wave system

Theorem

Given $T > 0$, suppose that:

1. (ω, T, p_{d_i}) *satisfies GCC,* $i = 1, 2$.
2. *Compatibility conditions.*
3. *The Kalman operator* $\mathcal{K} = [-D\Delta + A|\hat{b}]$ *satisfies the operator Kalman rank condition, i.e.* $Ker(\mathcal{K}^*) = \{0\}$.

Then the system (CWS) *is exactly controllable.*

Remark

As for compatibility conditions, for example, in the simple model (M2), $(u_1, u_2, u_3) \in H^4_\Omega \times H^2_\Omega \times H^1_\Omega$, *we have*

$$(-\Delta)^2 u_1 + \Delta u_2 \in H^1_0.$$

Outline for the proof

We prove the above theorem within three steps.

1. At first, we simplify the system (CWS), using a Brunovskỳ normal form. Based on the equivalent Kalman condition, we prove the exact controllability for the simplified system.
2. At the second step, we use the iteration schemes to obtain the compatibility conditions associated with the coupling structure. Therefore, we prepare the appropriate state spaces.
3. In the final step, we use HUM to derive the observability inequality and then follow the similar procedure. At last, the unique continuation property is given by the Kalman rank condition.

Some Comments

Remark

For the simple example (M2)*, we are also motiveted by Dehman-Le Rousseau-Léautaud 14', in which they considered two wave equations in different speeds in a compact manifold. For two equations, the compatibility conditions are trivial.*

Remark

We mainly consider the wave system in different speeds. While for the wave system in the same speed, one can refer to Alabau-Boussouira-Léautaud 13', Dehman-Le Rousseau-Léautaud 14', Cui-Laurent-Wang 20',etc. for different approaches to obtain the controllability.

Some further perspectives

As we have shown before, we can deal with several different types of coupling.

1. Different speeds, coupled by the control function;
2. The same speed, coupled by the states;
3. Different speeds, only coupled by different speed part.

A natural question is what would happen if we combining these types, for example:

$$\begin{cases} (\partial_t^2 - d_1\Delta)U_1 + A_{11}U_1 + A_{12}U_2 = B_1 f \mathbb{1}_\omega(x)\mathbb{1}_{[0,T]}(t), \\ (\partial_t^2 - d_2\Delta)U_1 + A_{21}U_1 + A_{22}U_2 = B_2 f \mathbb{1}_\omega(x)\mathbb{1}_{[0,T]}(t). \end{cases}$$

One can expect two kinds of difficulties. The first one is how to get a proper space associated with the coupling. The other one is the propagation argument near the boundary.

Thank you for your attention!